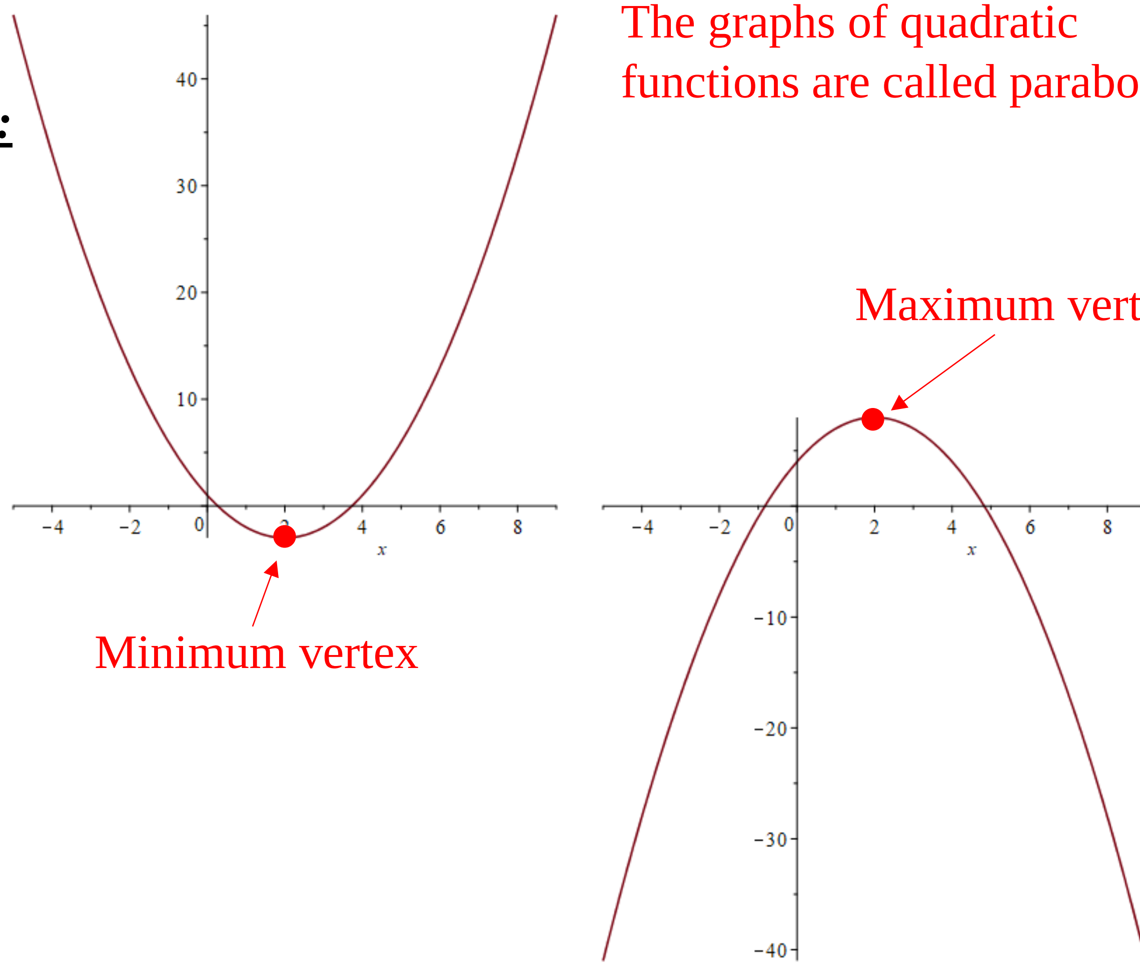
Review of Quadratic Functions:

The graphs of quadratic functions are called parabolas.



General Form:

$f(x) = ax^2 + bx + c \; ; a \neq 0$

Standard Form:

$f(x) = a(x-h)^2 + k \; ; a \neq 0$

Intercept Form:

$f(x) = a(x-x_1)(x-x_2) \; ; a \neq 0, x_1$ and x_2 are real numbers

<u>**General Form:**</u>

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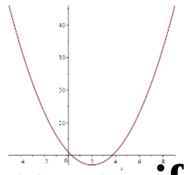
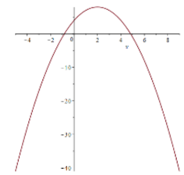
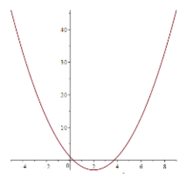


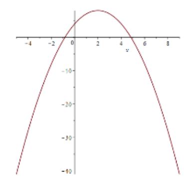


If $a > 0$**, the parabola opens up; if** $a < 0$**, the parabola opens down. The *y*-intercept is** c**.**

<u>**Standard Form:**</u>

$f(x) = a(x-h)^2 + k \; ; a \neq 0$



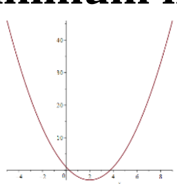


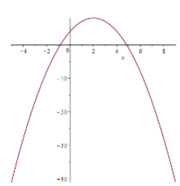
The vertex is at (h,k) **and it's a minimum vertex if** $a > 0$ **and a maximum vertex if** $a < 0$**.**

<u>**Intercept Form:**</u>

$f(x) = a(x-x_1)(x-x_2) \; ; a \neq 0, x_1$ and x_2 are real numbers

x_1 and x_2 **are the *x*-intercepts, and the *x*-coordinate of the vertex is** $\frac{x_1 + x_2}{2}$**. The vertex is a minimum if** $a > 0$ **and a maximum if** $a < 0$**.**



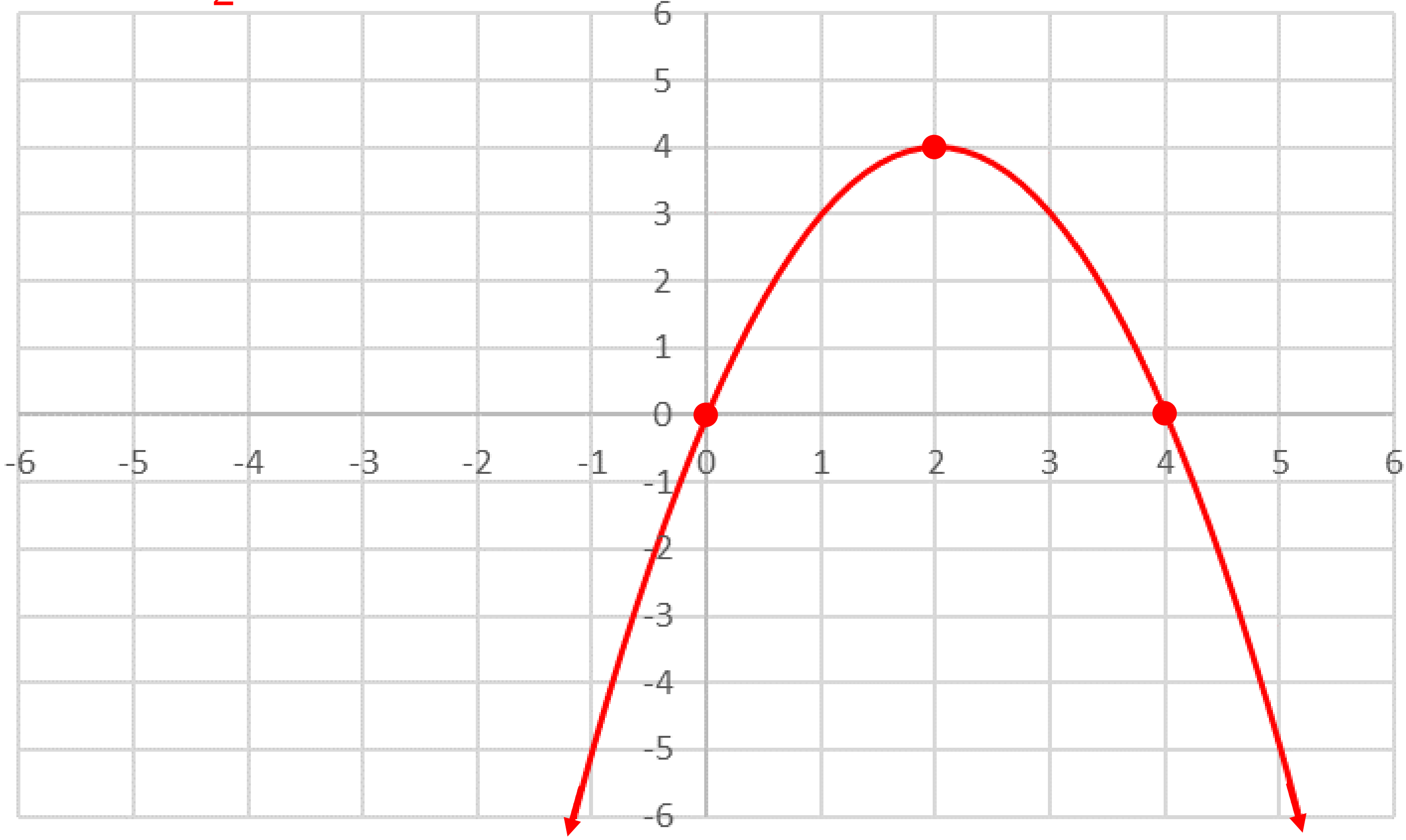


Graph the following quadratic functions. Indicate the vertex and all the intercepts.

1. $f(x) = -x^2 + 4x$ ***{Convert to intercept form by factoring out $-x$.}***

$= -x(x-4)$

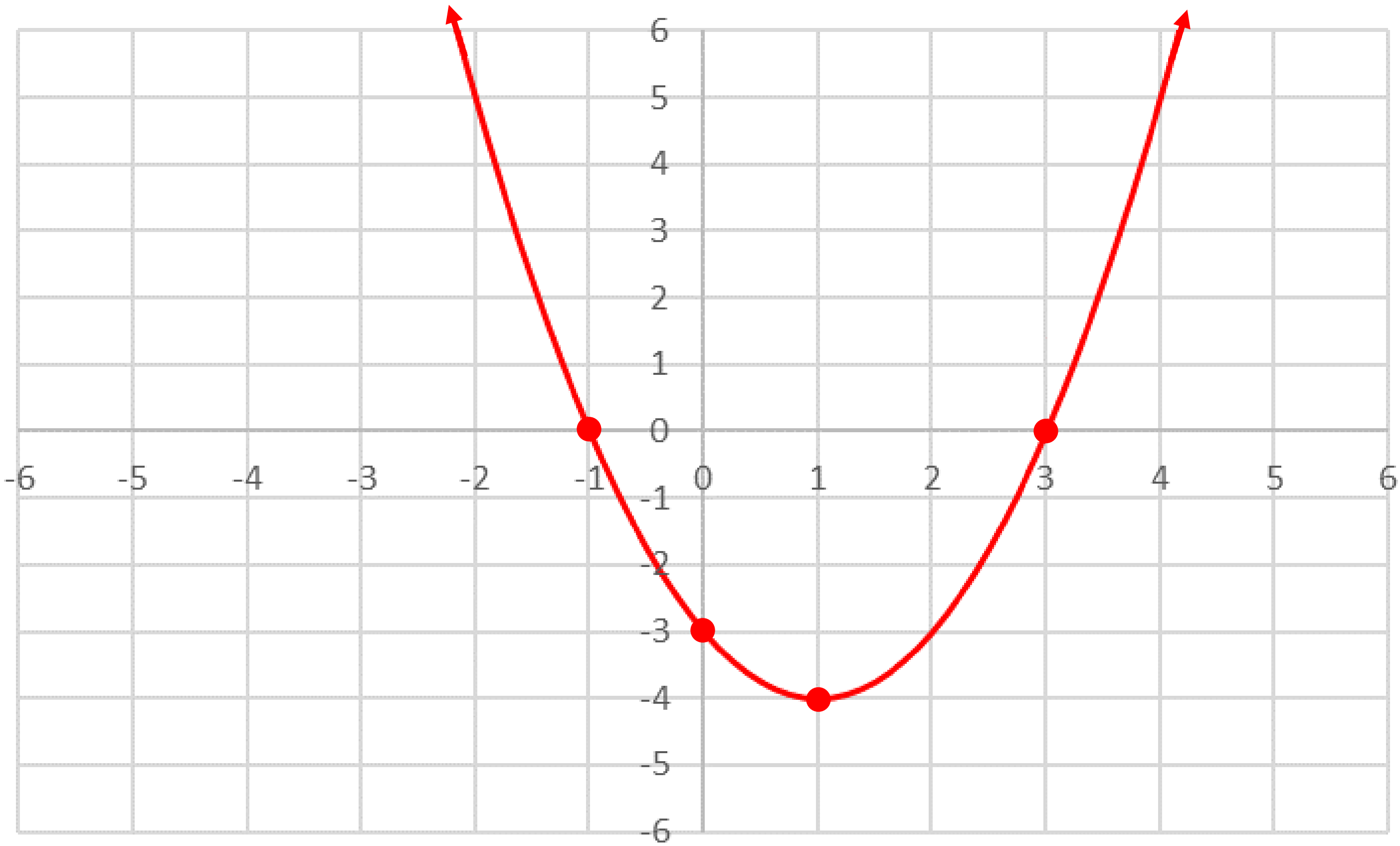
The x-coordinate of the vertex is the average of the two x-intercepts: $\frac{0+4}{2} = 2$ and $f(2) = -4 + 8 = 4$, $f(0) = 0$.



2. $f(x) = x^2 - 2x - 3$ ***{Convert to intercept form by factoring the trinomial.}***

$= (x-3)(x+1)$ The x-coordinate of the vertex is the average of the two x-intercepts:

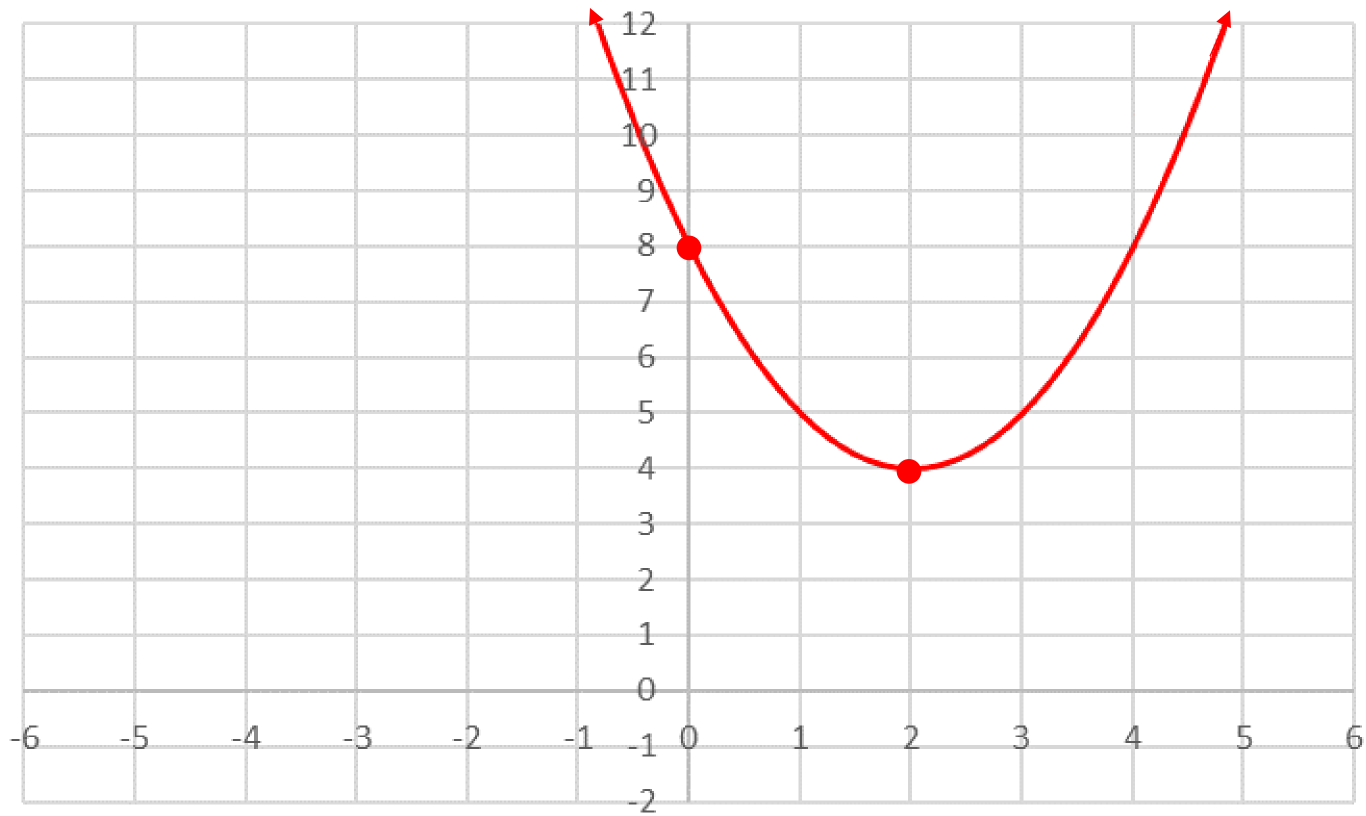
$\frac{3+(-1)}{2} = 1$ and $f(1) = 1 - 2 - 3 = -4,\ f(0) = -3.$



3. $f(x) = (x-2)^2 + 4$

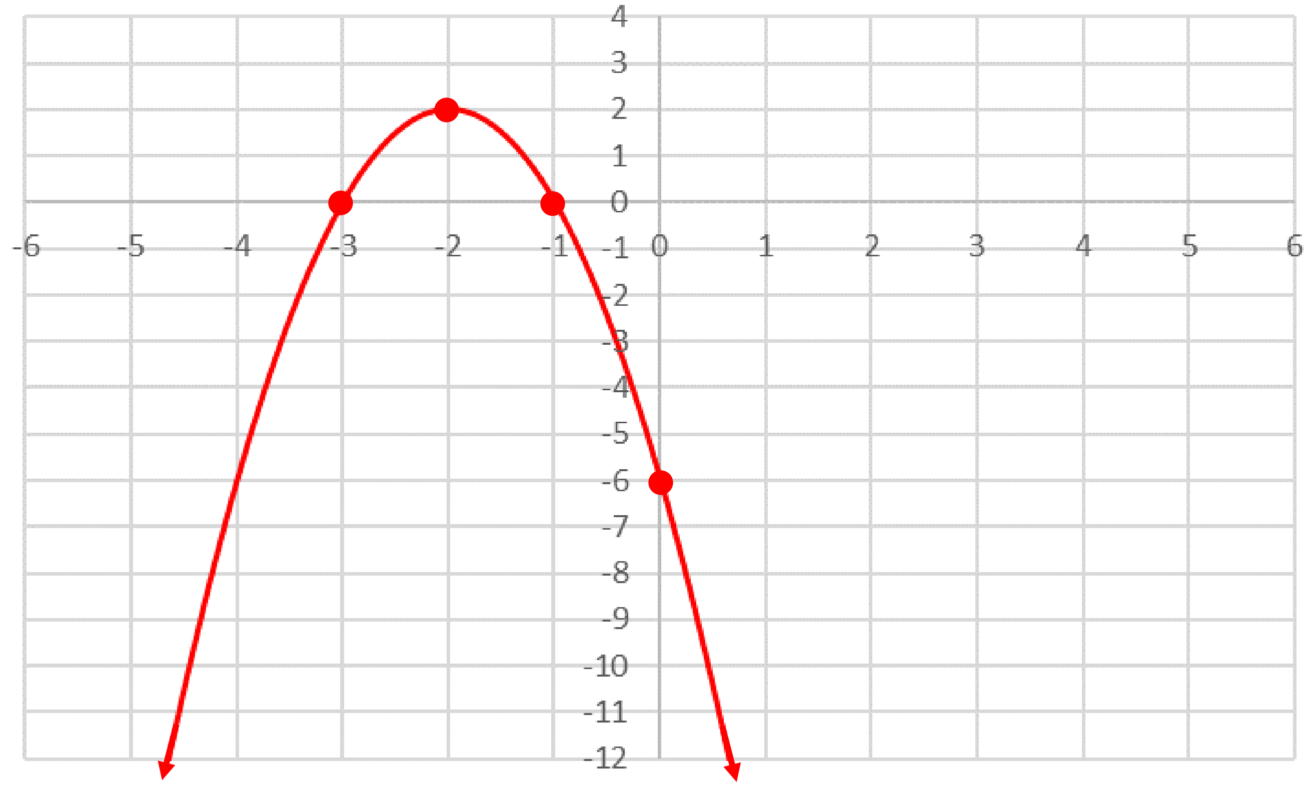
{It's in standard form.}

$f(0) = 4 + 4 = 8$



4. $f(x) = -2(x+2)^2 + 2$ ***{It's in standard form.}***

$2(x+2)^2 = 2 \Rightarrow (x+2)^2 = 1 \Rightarrow x+2 = \pm 1 \Rightarrow x = -1, -3$ and $f(0) = -8+2 = -6$.



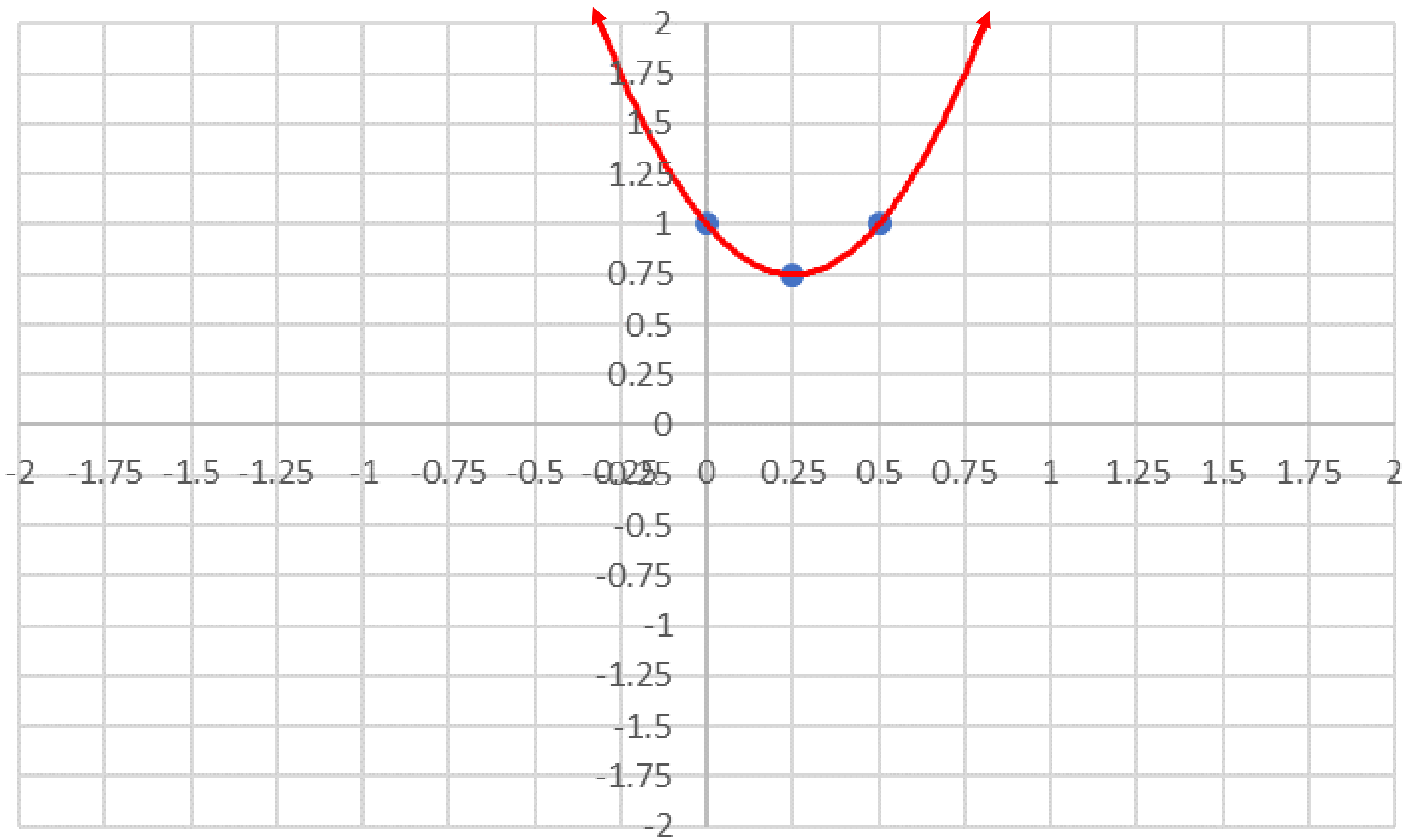
5. $f(x) = 4x^2 - 2x + 1$ ***{Convert to standard form by completing the square.}***

$$= 4\left(x^2 - \tfrac{1}{2}x\right) + 1$$

$$= 4\left(x^2 - \tfrac{1}{2}x + \tfrac{1}{16}\right) + 1 - \tfrac{1}{4}$$

$$= 4\left(x - \tfrac{1}{4}\right)^2 + \tfrac{3}{4}$$

$$f(0) = 1$$



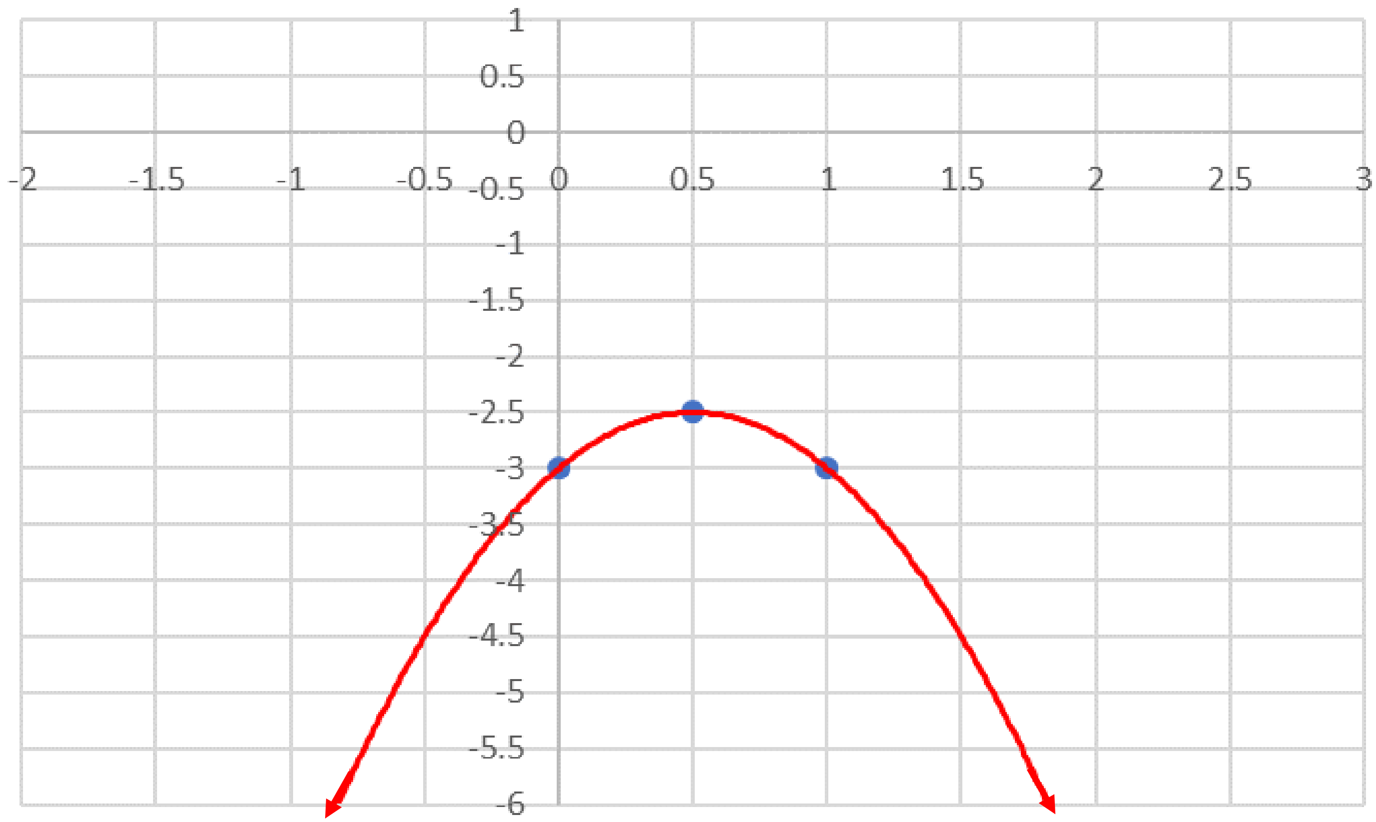
6. $f(x) = -2x^2 + 2x - 3$ ***{Convert to standard form by completing the square.}***

$$= -2\left(x^2 - x\right) - 3$$

$$= -2\left(x^2 - x + \tfrac{1}{4}\right) - 3 + \tfrac{1}{2}$$

$$= -2\left(x - \tfrac{1}{2}\right)^2 - \tfrac{5}{2}$$

$$f(0) = -3$$



Finding Formulas for Quadratic Functions:

1. The vertex of the parabola is $(1,2)$, and it passes through the point $(3,0)$.

$$\left\{ f(x) = a(x-h)^2 + k \right\}$$

$$f(x) = a(x-1)^2 + 2$$

$$\Rightarrow 0 = a(3-1)^2 + 2$$

$$\Rightarrow 0 = 4a + 2$$

$$\Rightarrow a = -\frac{1}{2}$$

$$\Rightarrow \boxed{f(x) = -\frac{1}{2}(x-1)^2 + 2}$$

2. The *x*-intercepts are 5 and -3, and the graph passes through the point $(0,-4)$.

$$\{f(x) = a(x - x_1)(x - x_2)\}$$

$$f(x) = a(x-5)(x+3)$$

$$\Rightarrow -4 = a(0-5)(0+3)$$

$$\Rightarrow -4 = -15a$$

$$\Rightarrow a = \frac{4}{15}$$

$$\Rightarrow \boxed{f(x) = \frac{4}{15}(x-5)(x+3)}$$

3. The graph passes through the points $(0,1)$, $(1,2)$, and $(-1,4)$.

$$\left\{ f(x) = ax^2 + bx + c \right\}$$

$$f(x) = ax^2 + bx + c$$

Substitute the x and y coordinates into the formula for the function.

$$\underline{1 = c}$$

$$2 = a + b + c$$

$$4 = a - b + c$$

Substitute 1 for c in the last two equations.

$$\Rightarrow 1 = a + b$$

$$3 = a - b$$

Add the two equations together.

$$\Rightarrow 4 = 2a$$

$$\Rightarrow \underline{a = 2}, \underline{b = -1}$$

Plug in the values of a, b, and c into the function formula.

$$\Rightarrow \boxed{f(x) = 2x^2 - x + 1}$$

4. The graph passes through the points $(1,2)$ and $(5,2)$, and the minimum value of the function is -4.

$$\left\{ f(x) = a(x-h)^2 + k \right\}$$

Since the two points on the graph have the same *y*-coordinate, by symmetry the average of their *x*-coordinates must be the *x*-coordinate of the vertex. The minimum value would have to be the *y*-coordinate of the vertex.

$$f(x) = a(x-3)^2 - 4$$

$$\Rightarrow 2 = a(1-3)^2 - 4$$

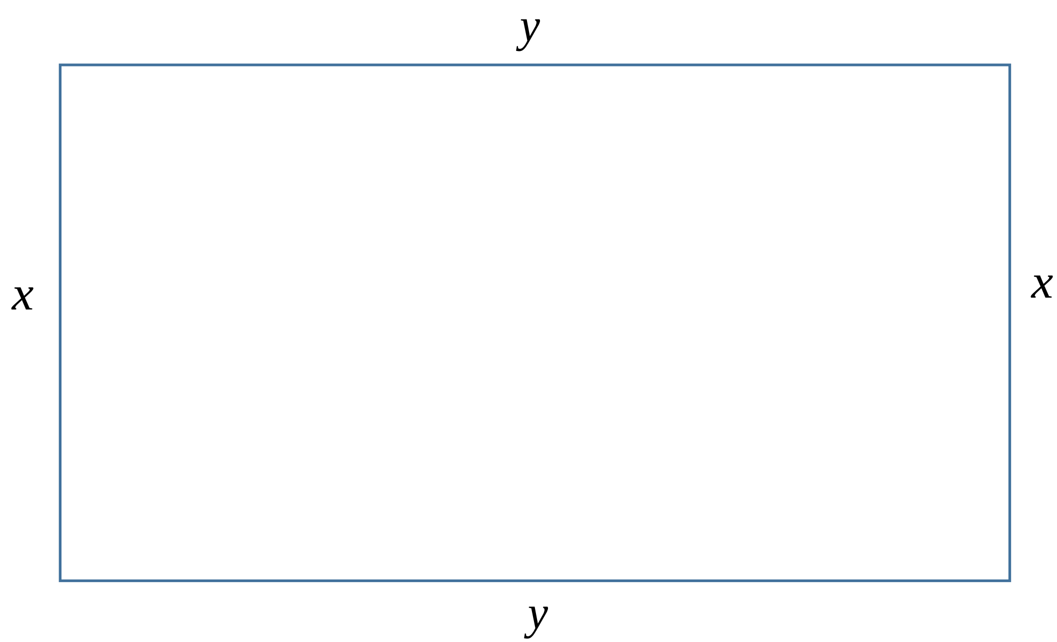
$$\Rightarrow 2 = 4a - 4$$

$$\Rightarrow a = \frac{3}{2}$$

$$\Rightarrow \boxed{f(x) = \frac{3}{2}(x-3)^2 - 4}$$

Word Problems:

1. Joe has 3,000 feet of fence available to enclose a rectangular field.



$$2x+2y=3000$$

$$x+y=1500$$

$$y=1500-x$$

a) Express the enclosed area, *A*, as a function of *x*.

$$A = xy$$

Replace the y with $1500 - x$.

$$\Rightarrow A = x(1500 - x)$$

$$\Rightarrow \boxed{A(x) = x(1500 - x)}$$

b) Determine the domain of the function, $A(x)$.

Both of the dimensions of the rectangle must be positive, so

$$x > 0, 1500 - x > 0$$

$$\Rightarrow x > 0 \text{ and } x < 1500$$

$$\Rightarrow \boxed{0 < x < 1500 \quad \text{or} \quad (0, 1500)}$$

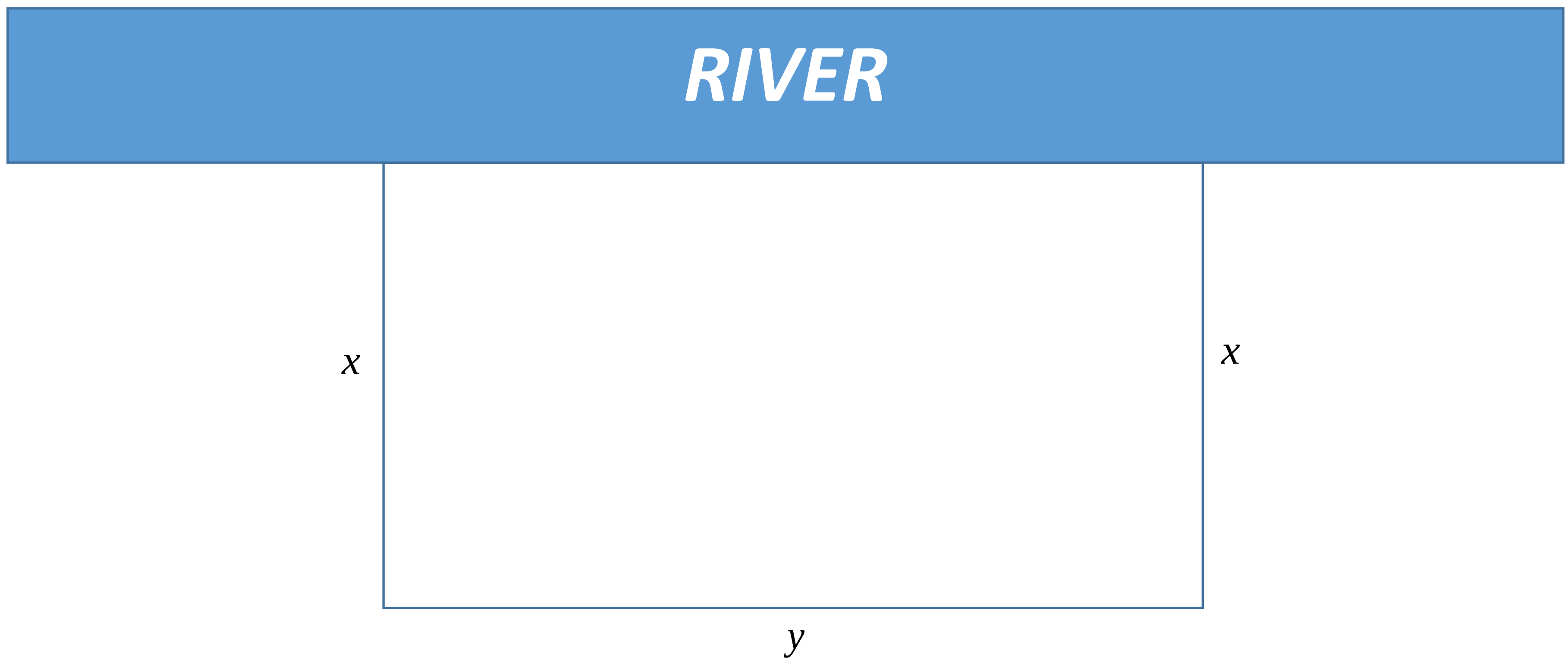
c) For what value of *x* is the enclosed area the largest?

The graph of the quadratic function that represents the enclosed area takes on the value zero at $x = 0, 1500$, so the value of x that corresponds to the maximum enclosed area is the average of these two values. So $x = \boxed{750\,ft}$.

d) What is the value of the largest enclosed area?

$$A(750) = 750 \cdot 750 = \boxed{562{,}500\,ft^2}$$

2. A farmer with 2,000 yards of fence wants to enclose a rectangular field that borders on a straight river-so he'll only need fence on three sides of the field.



$$2x + y = 2000$$

$$y = 2000 - 2x$$

a) Express the enclosed area, *A*, as a function of *x*.

$$A = xy$$

Replace y with $2000 - 2x$.

$$\Rightarrow A = x(2000 - 2x)$$

$$\Rightarrow A = 2x(1000 - x)$$

$$\Rightarrow \boxed{A(x) = 2x(1000 - x)}$$

b) Determine the domain of the function, $A(x)$.

Both of the dimensions of the rectangle must be positive, so

$$x > 0, 2000 - 2x > 0$$

$$\Rightarrow x > 0 \text{ and } 2x < 2000$$

$$\Rightarrow x > 0 \text{ and } x < 1000$$

$$\Rightarrow \boxed{0 < x < 1000 \quad \text{or} \quad (0,1000)}$$

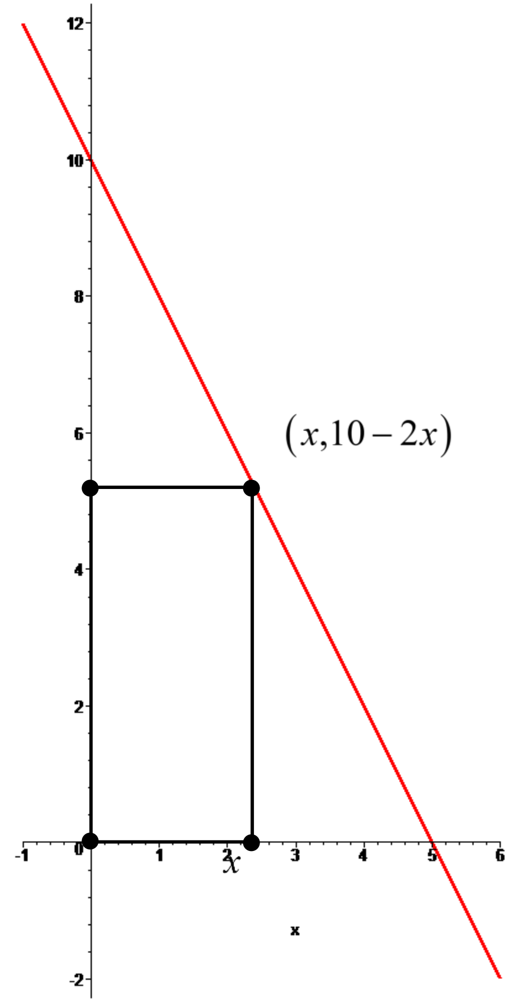
c) For what value of *x* is the enclosed area the largest?

The graph of the quadratic function that represents the enclosed area takes on the value zero at $x = 0{,}1000$, so the value of x that corresponds to the maximum enclosed area is the average of these two values. So $x = \boxed{500\,yds}$.

d) What is the value of the largest enclosed area?

$$A(500) = 1000 \cdot 500 = \boxed{500{,}000\,yd^2}$$

3. A rectangle in the first quadrant has one vertex on the line $y = 10 - 2x$**, another at the origin, one on the positive** ***x*****-axis, and one on the positive** ***y*****-axis.** ***(See the figure.)***



a) Express the area *A* of the rectangle as a function of *x*.

$$A = xy$$

Replace y with $10-2x$.

$$\Rightarrow A = x(10-2x)$$

$$\Rightarrow A = 2x(5-x)$$

$$\Rightarrow \boxed{A(x) = 2x(5-x)}$$

b) What's the domain of $A(x)$?

In order to have such rectangle, $\boxed{0 < x < 5 \text{ or } (0,5)}$

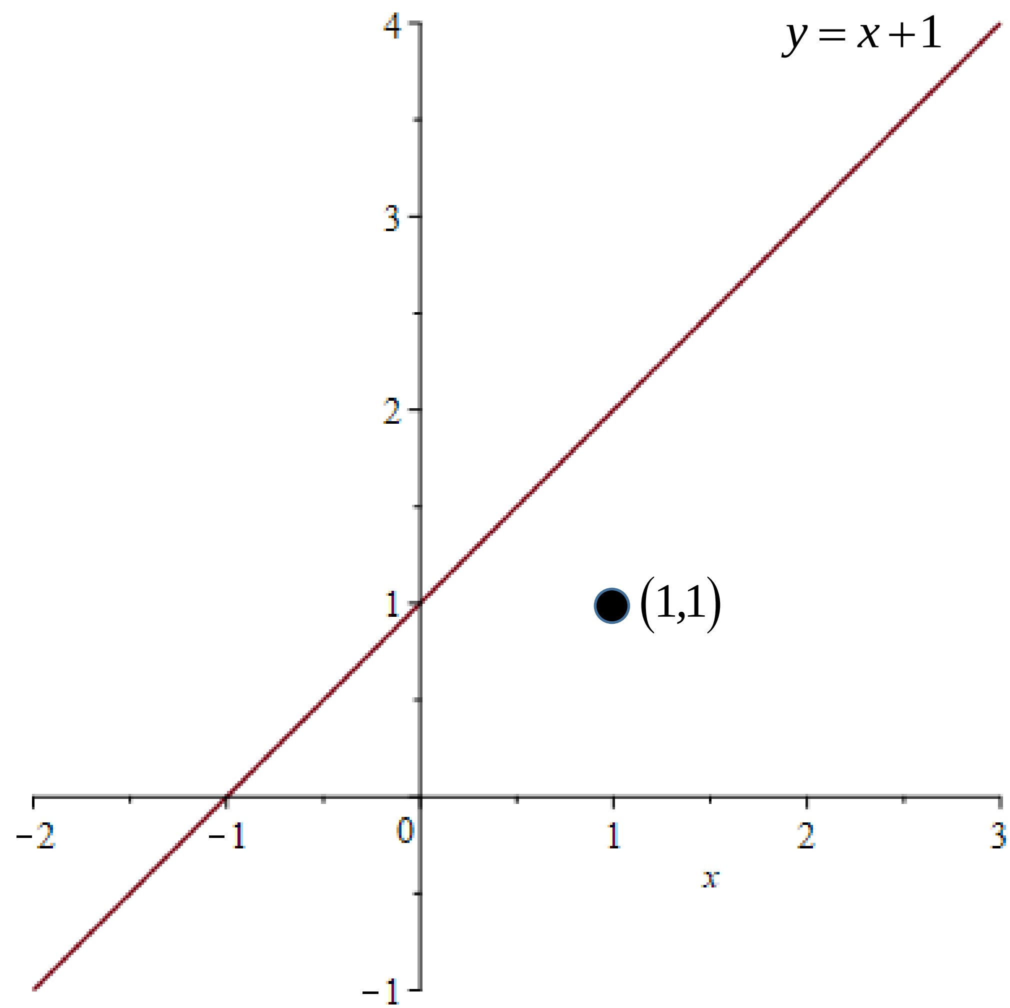
c) What value of *x* produces the maximum area?

The graph of the quadratic function that represents the enclosed area takes on the value zero at $x = 0, 5$, so the value of x that corresponds to the maximum enclosed area is the average of these two values. So $x = \boxed{\frac{5}{2} units}$.

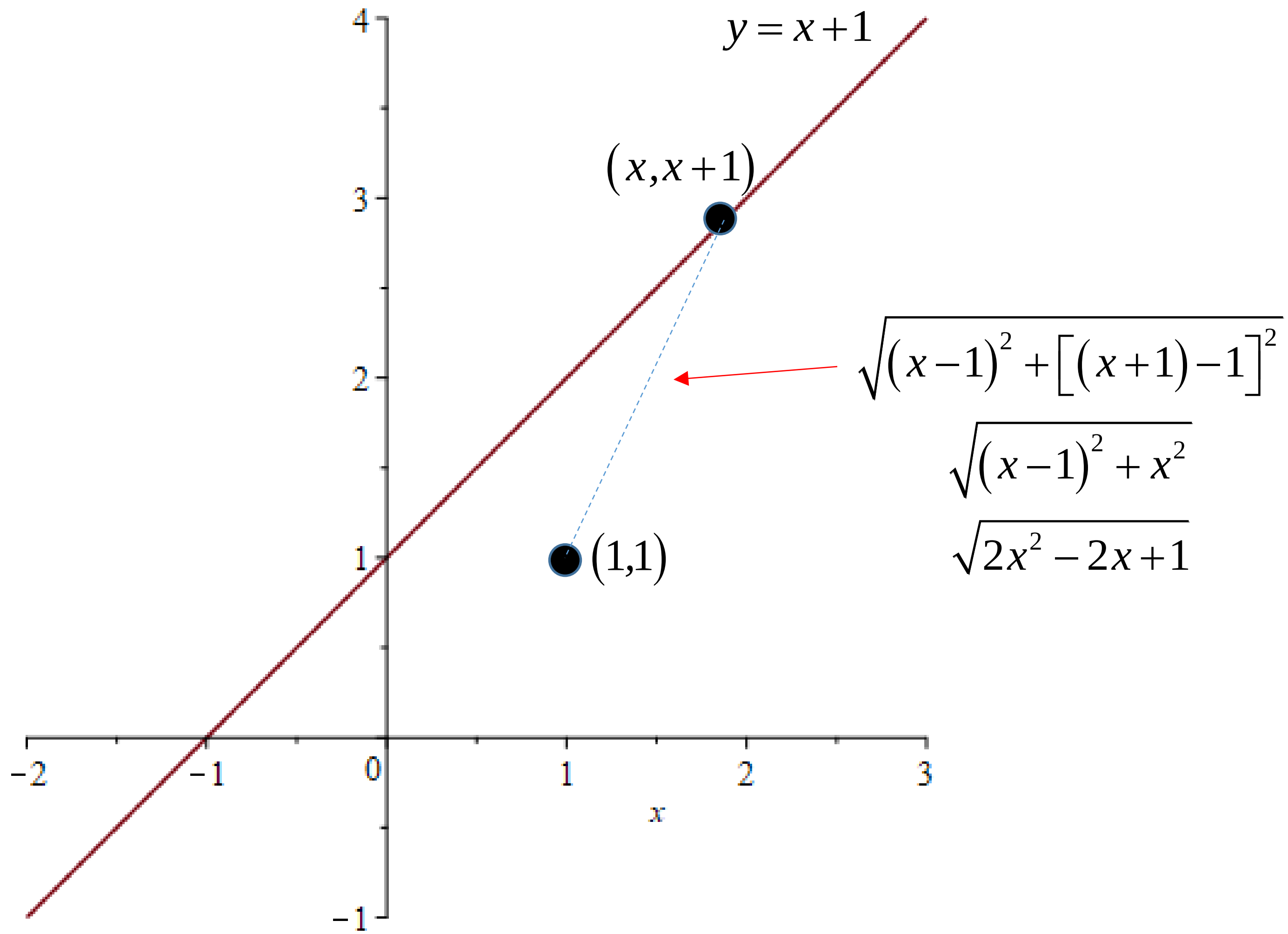
d) What is the maximum area?

$$A\left(\frac{5}{2}\right) = 5 \cdot \frac{5}{2} = \boxed{\frac{25}{2} unit^2}$$

4. Find the point on the line $y = x + 1$ that is closest to the point $(1,1)$.



a) Express the distance from a point on the line $(x, x+1)$ to the point $(1,1)$ as a function of *x*.



b) Express the square of the distance as a nice quadratic function.

$$D = \left(\sqrt{2x^2 - 2x + 1}\right)^2$$

$$\Rightarrow \boxed{D(x) = 2x^2 - 2x + 1}$$

c) What's the domain of the function?

There are no restrictions on x, so the domain is $(-\infty, \infty)$.

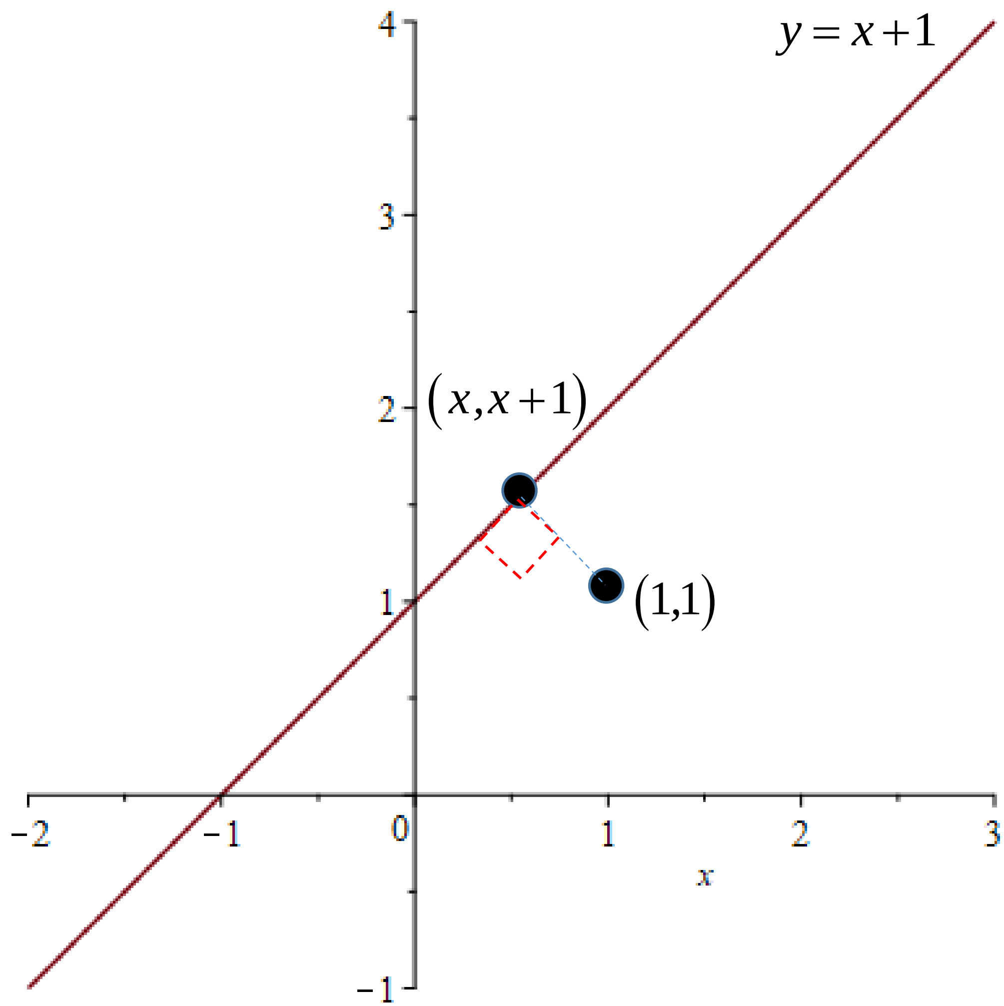
d) Use the quadratic function to find the closest point on the line.

Complete the square to get standard form.

$$\begin{aligned} D(x) &= 2x^2 - 2x + 1 \\ &= 2(x^2 - x) + 1 \\ &= 2\left(x^2 - x + \tfrac{1}{4}\right) + 1 - \tfrac{1}{2} \\ &= 2\left(x - \tfrac{1}{2}\right)^2 + \tfrac{1}{2} \end{aligned}$$

So the closest point has an x-coordinate of $\frac{1}{2}$, and therefore, the closest point is $\boxed{\left(\frac{1}{2}, \frac{3}{2}\right)}$.

e) An alternative method using geometry.



When you're at the closest point, the angle formed must be a right angle, so the slope from $(1,1)$ to $(x,x+1)$ would have to be the negative reciprocal of the slope of the line $y=x+1$.

$$\{\text{slope from } (1,1) \text{ to } (x,x+1)\}\frac{x+1-1}{x-1}=-1\{\text{negative reciprocal of the slope}\}$$

$$\frac{x+1-1}{x-1}=-1$$

$$\Rightarrow \frac{x}{x-1}=-1$$

Cross-multiply.

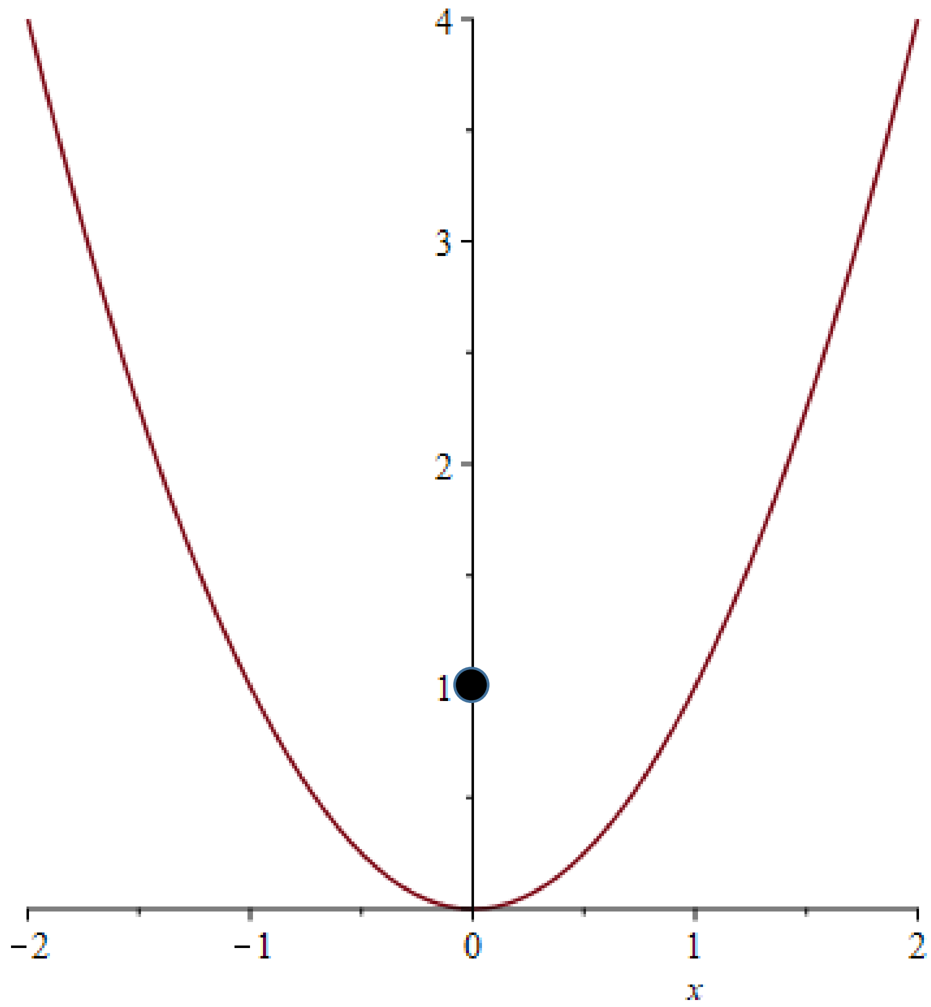
$$\Rightarrow x=-x+1$$

$$\Rightarrow 2x=1$$

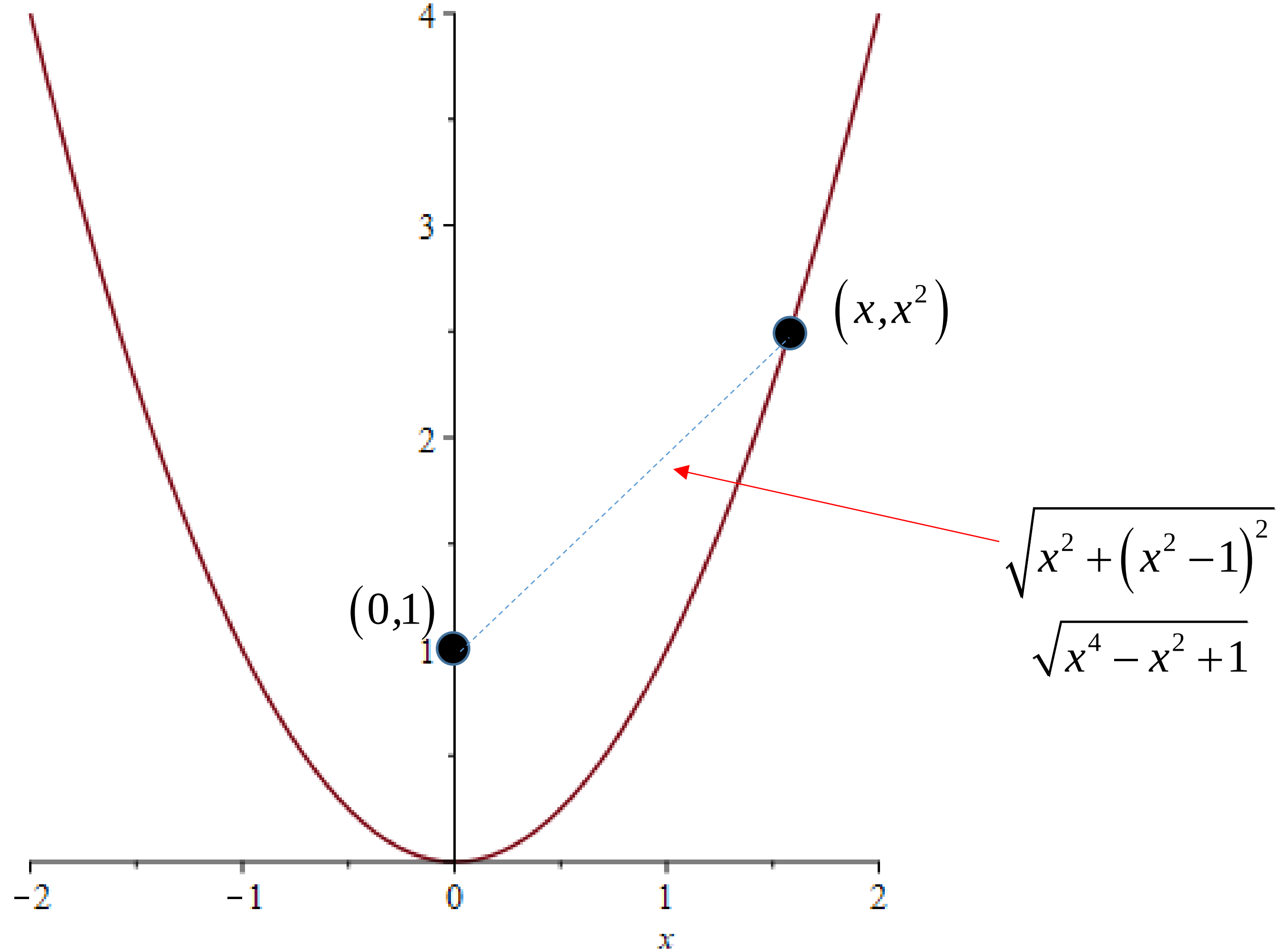
$$\Rightarrow x=\frac{1}{2}$$

So we get the same point that we got in the previous approach with much less effort.

5. Find the point(s) on the parabola $y = x^2$ closest to the point $(0,1)$.



a) Express the distance from a point on the parabola (x, x^2) to the point $(0,1)$ as a function of *x*.



b) Express the square of the distance as a nice quadratic function in x^2.

$$D = \left(\sqrt{x^4 - x^2 + 1}\right)^2$$

$$\Rightarrow \boxed{D(x) = x^4 - x^2 + 1}$$

c) Use completing the square to find the point(s) on the parabola closest to $(0,1)$.

$$\begin{aligned} D(x) &= x^4 - x^2 + 1 \\ &= \left(x^4 - x^2 + \tfrac{1}{4}\right) + 1 - \tfrac{1}{4} \\ &= \left(x^2 - \tfrac{1}{2}\right)^2 + \tfrac{3}{4} \end{aligned}$$

This will take on its smallest value when $x^2 - \tfrac{1}{2} = 0 \Rightarrow x = \pm\sqrt{\tfrac{1}{2}}$. So the closest points are

$$\boxed{\left(\sqrt{\frac{1}{2}}, \frac{1}{2}\right) \text{ and } \left(-\sqrt{\frac{1}{2}}, \frac{1}{2}\right)}.$$

When you take Calculus, you will be introduced to methods for solving problems like this that are a little quicker.