Review of Reduced Rational Functions:

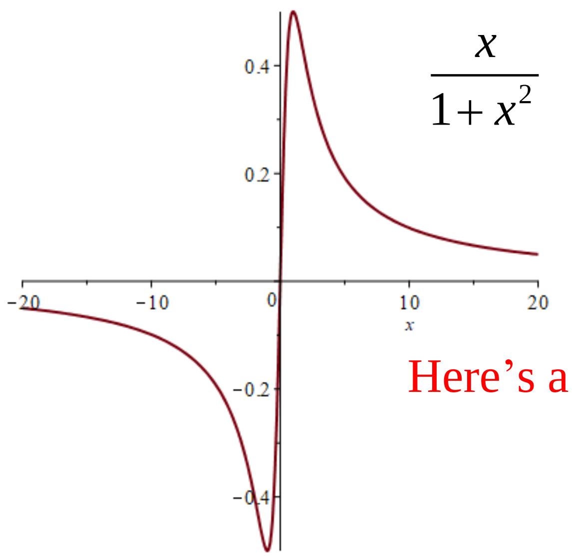
$f(x) = \dfrac{p(x)}{q(x)}$, **where** $p(x)$ **and** $q(x)$ **have no common real zeros.** ***If they have common imaginary zeros, then the corresponding common factors should be cancelled out.***

Asymptotes:

Lines that the graph of a rational function approach.

Horizontal:

If the degree of $p(x)$ **is less than the degree of** $q(x)$**, then** $y = 0$ **(*x*-axis) is the horizontal asymptote.**



$$\frac{x}{1+x^2}$$

For $|x|$ large, $\dfrac{x}{1+x^2} = \dfrac{\frac{x}{x^2}}{\frac{1}{x^2} + \frac{x^2}{x^2}} = \dfrac{\overbrace{\frac{1}{x}}^{0}}{\underbrace{\frac{1}{x^2}}_{0} + 1} \approx \dfrac{0}{0+1} = 0$

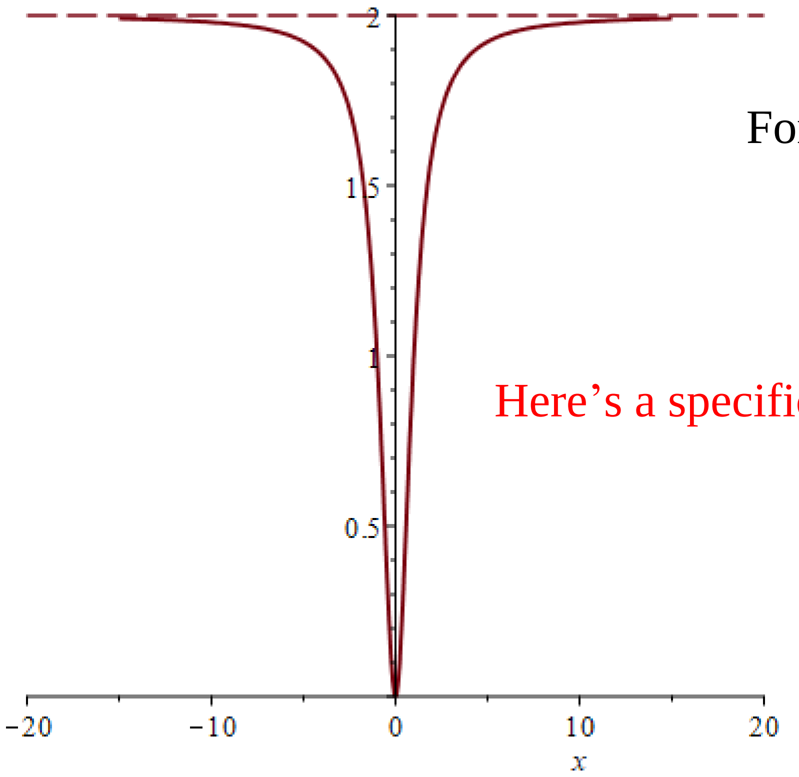
Divide everything by the highest power of x in the denominator.

Here’s a specific example.

If the degree of $p(x)$ is equal to the degree of $q(x)$, then

$$y = \frac{\text{leading coefficient of } p(x)}{\text{leading coefficient of } q(x)}$$ **is the horizontal asymptote.**

$$\frac{2x^2}{1+x^2}$$



For $|x|$ large, $\dfrac{2x^2}{1+x^2} = \dfrac{\frac{2x^2}{x^2}}{\frac{1}{x^2}+\frac{x^2}{x^2}} = \dfrac{2}{\underbrace{\frac{1}{x^2}}_{0}+1} \approx \dfrac{2}{0+1} = 2$

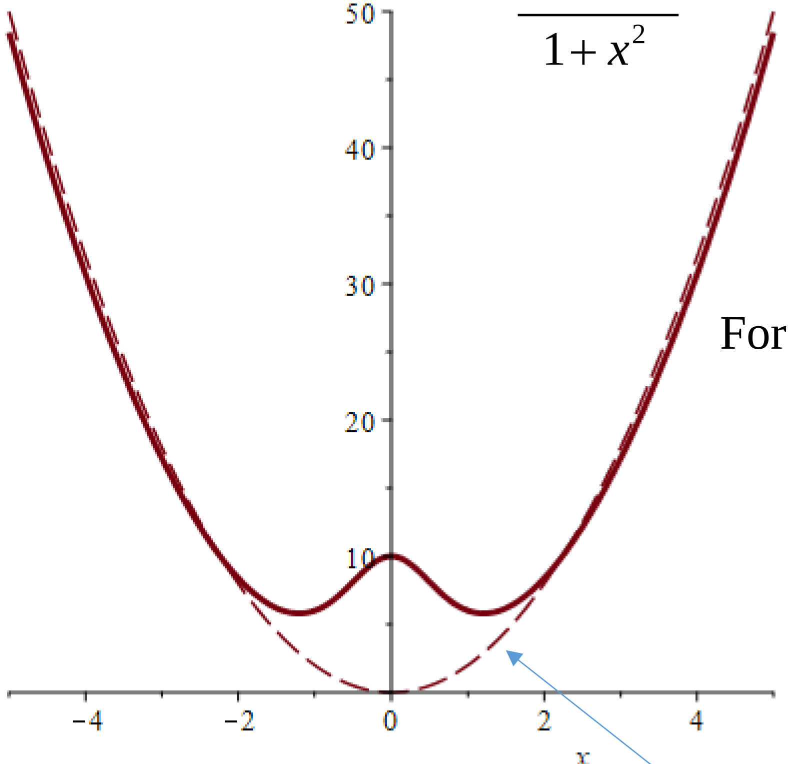
Here's a specific example.

Divide everything by the highest power of x in the denominator.

If the degree of $p(x)$ is greater than the degree of $q(x)$, then there is no horizontal asymptote. The end behavior is same as the end behavior of the polynomial

$$\frac{\text{highest degreed term in } p(x)}{\text{highest degreed term in } q(x)}.$$

$$\frac{2x^4+10}{1+x^2}$$



Here's a specific example.

$$\text{For } |x| \text{ large, } \frac{2x^4+10}{1+x^2} = \frac{\frac{2x^4}{x^2}+\frac{10}{x^2}}{\frac{1}{x^2}+\frac{x^2}{x^2}} = \frac{2x^2+\frac{10}{x^2}}{\frac{1}{x^2}+1} \approx \frac{2x^2+0}{0+1} = 2x^2$$

Divide everything by the highest power of x in the denominator.

End behavior is the same as $\frac{2x^4}{x^2} = 2x^2$.

Examples:

1. $f(x)=\dfrac{2x^3+x-1}{3x^3+x^2}$ **H.A.:** $y=\dfrac{2}{3}$

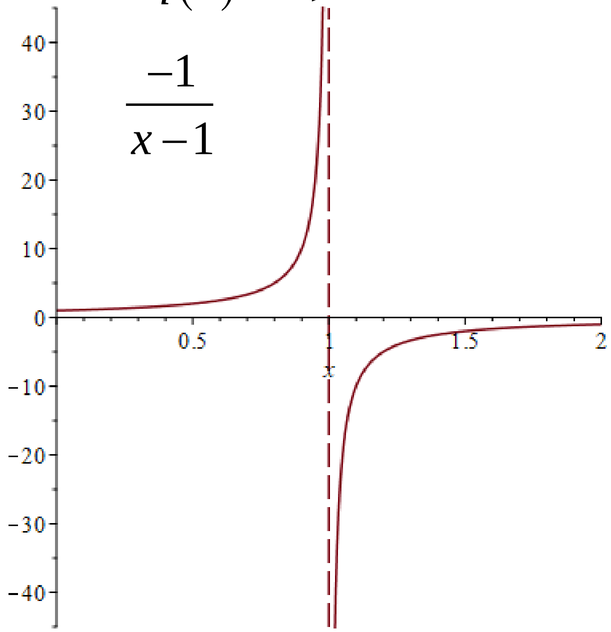
2. $f(x)=\dfrac{2x^2+x-1}{3x^3+x^2}$ **H.A.:** $y=0$

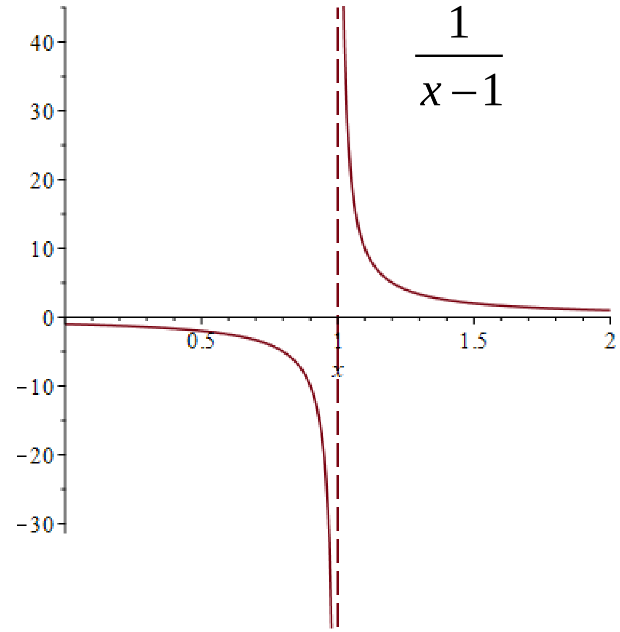
3. $f(x)=\dfrac{2x^6+x-1}{3x^3+x^2}$ **H.A.:** none, but the end behavior is the same as

$$y=\frac{2}{3}x^3$$

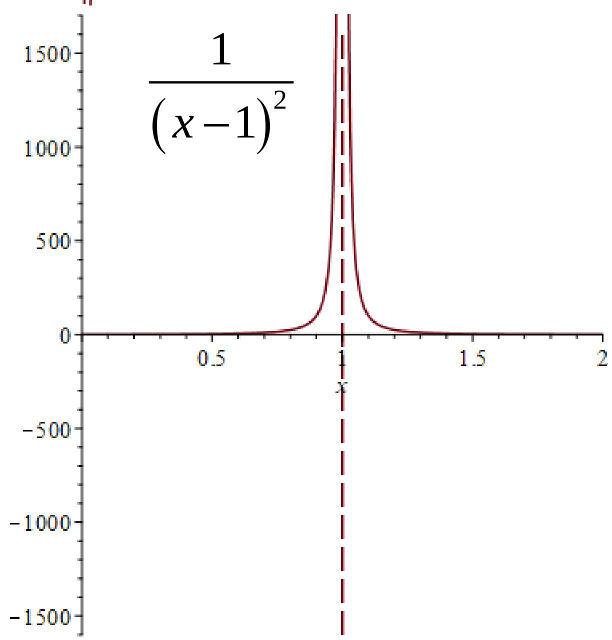
Vertical:

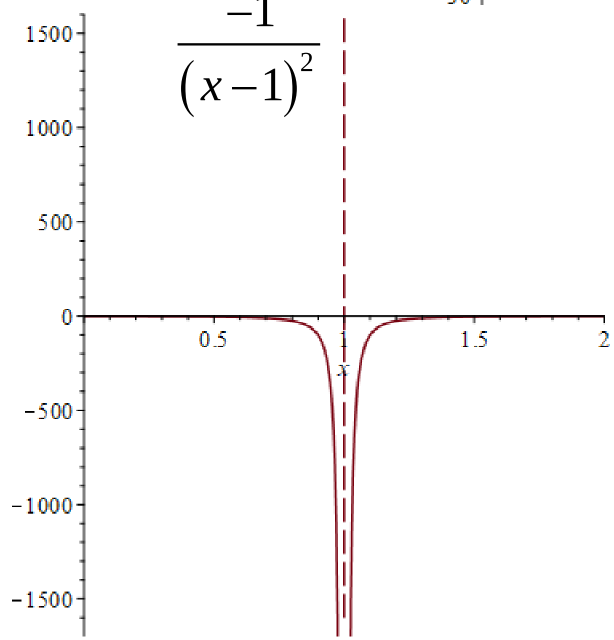
If $q(c)=0$, for c a real number, then $x=c$ is a vertical asymptote.





Here are some examples.





Examples:

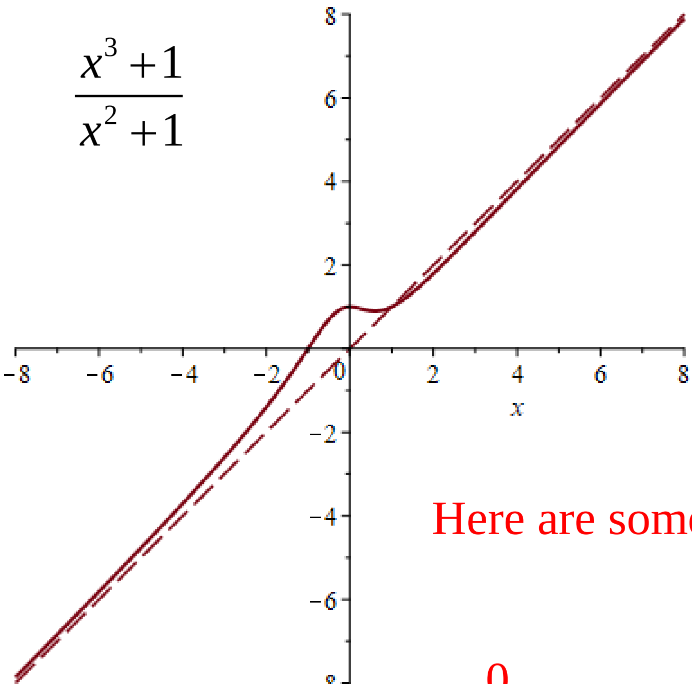
1. $f\left(x\right)=\dfrac{3x+5}{x-6}$ **V.A.:** $x=6$

2. $f\left(x\right)=\dfrac{x^{3}+1}{x^{2}-5x-14}=\dfrac{x^{3}+1}{\left(x-7\right)\left(x+2\right)}$ **V.A.:** $x=7, x=-2$

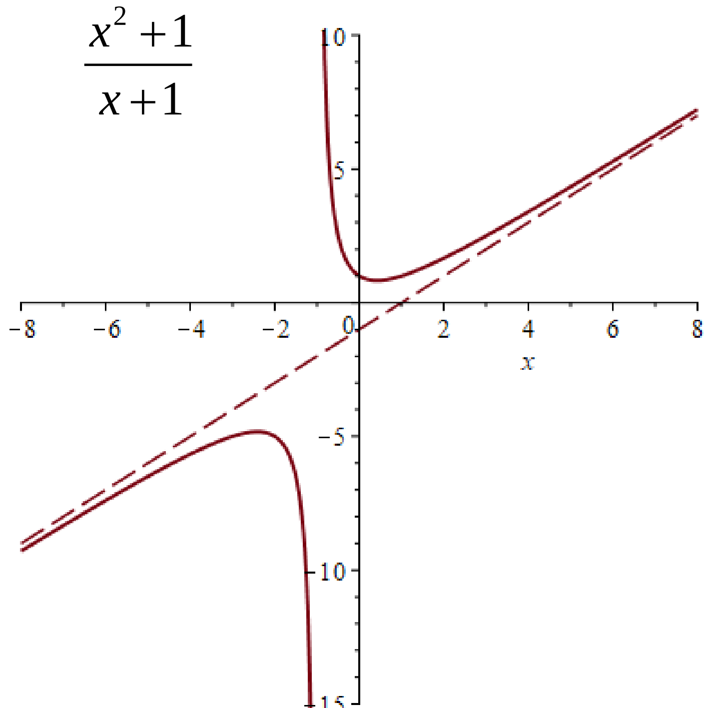
3. $f\left(x\right)=\dfrac{x^{3}}{x^{4}-1}=\dfrac{x^{3}}{\left(x^{2}+1\right)\left(x-1\right)\left(x+1\right)}$ **V.A.:** $x=1, x=-1$

Slant/Oblique:

If the degree of $p(x)$ is 1 more than the degree of $q(x)$, then the rational function will have a slant/oblique asymptote. The equation of the slant asymptote can be determined using polynomial long division.



Here are some examples.



0

$$\frac{x^3+1}{x^2+1} = x - \frac{x-1}{x^2+1}$$

0

$$\frac{x^2+1}{x+1} = x-1+\frac{2}{x+1}$$

Examples:

1. $f(x)=\dfrac{2x^2+x-1}{x+2}$

$$
\begin{array}{r|l}
 & \quad 2x-3 \\
\hline
x+2 & \quad 2x^2+x-1 \\
 & -\left(2x^2+4x\right) \\
\hline
 & \quad -3x-1 \\
 & -\left(-3x-6\right) \\
\hline
 & \qquad 5
\end{array}
$$

S.A.: $y=2x-3$

2. $f(x)=\dfrac{x^3+8}{x^2-5x+6}$

$$
\begin{array}{r|l}
 & \quad x+5 \\
\hline
x^2-5x+6 & \quad x^3+8 \\
 & -\left(x^3-5x^2+6x\right) \\
\hline
 & \quad 5x^2-6x+8 \\
 & -\left(5x^2-25x+30\right) \\
\hline
 & \qquad 19x-22
\end{array}
$$

S.A.: $y=x+5$

<u>Sign Charts for Rational Functions:</u>

The behavior of a rational function in the vicinity of its vertical asymptotes can be determined by the sign of the function values. To make a sign chart for a rational function, draw a number line and locate the zeros of the numerator and label them with a *0*, since the function value is zero there. Locate the zeros of the denominator, and label them with a *u*, since the function is undefined at these values. Use what you know about the graphs of polynomial functions to determine the sign of the rational function on the intervals in between and on the edges.

Examples:

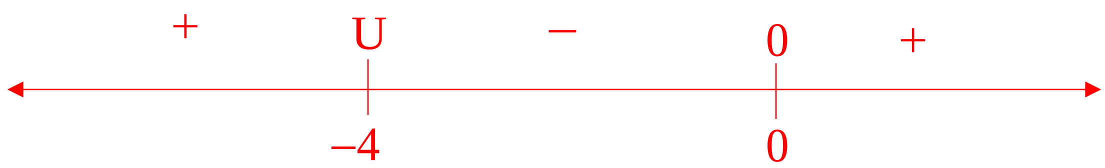
The arrows indicate the end behavior of the numerator and denominator, so we can quickly know the sign of the rational function left and right. In the middle we can use the fact that an odd exponent signals a sign change, while an even exponent signals no sign change.

1. $f\left(x\right)=\dfrac{3x}{x+4}$

The first thing to do is label the zeros of the numerator and denominator on a number line.

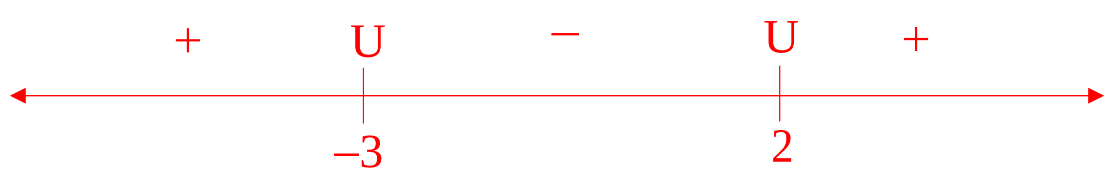
We can either choose a value larger than zero, a value between -4 and zero, and a value smaller than -4 to figure out the sign pattern of the rational function, or we can use what we know about polynomials. $3x$ and $x+4$ are eventually positive to the right(end behavior), so their ratio must be positive to the right.

Polynomial functions change sign(cross the *x*-axis) for odd exponents and don't change sign for even exponents. The exponent on *x* is odd, and the exponent on *x* + 4 is odd. This means that as we cross zero, the sign must change, and as we cross -4, the sign must change.

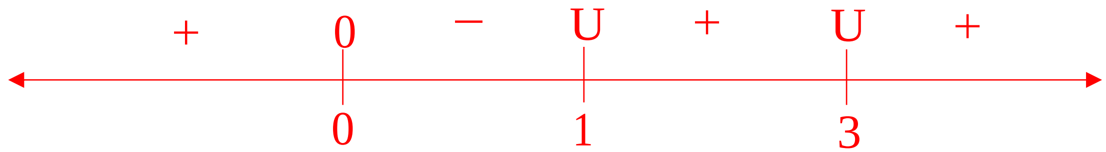


2. $f(x)=\dfrac{6}{x^2+x-6}$

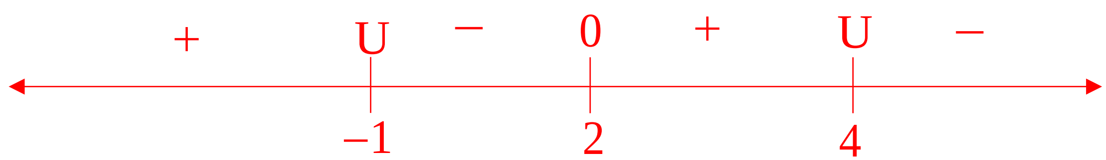
$$\frac{6}{x^2+x-6}=\frac{6}{(x+3)(x-2)}$$



3. $f(x)=\dfrac{3x}{(x-1)(x-3)^2}$



4. $f(x)=\dfrac{2-x}{(x+1)(x-4)}$



Sketching graphs of rational functions:

1. Find and draw the horizontal, vertical, and slant asymptotes as dashed lines.

2. Find and label the x-intercepts as points.

3. Find and label the y-intercept as a point.

4. Create the sign chart.

5. Connect all the dots in a reasonable manner using the sign chart as a guide.

The goal in sketching the graph of a rational function is to plot as few points as possible(the x and y intercepts), and use the x-intercept behavior and the behavior near the asymptotes to capture the qualitative behavior of the graph. Other than the y-intercept, the y-coordinates are not important.

Examples:

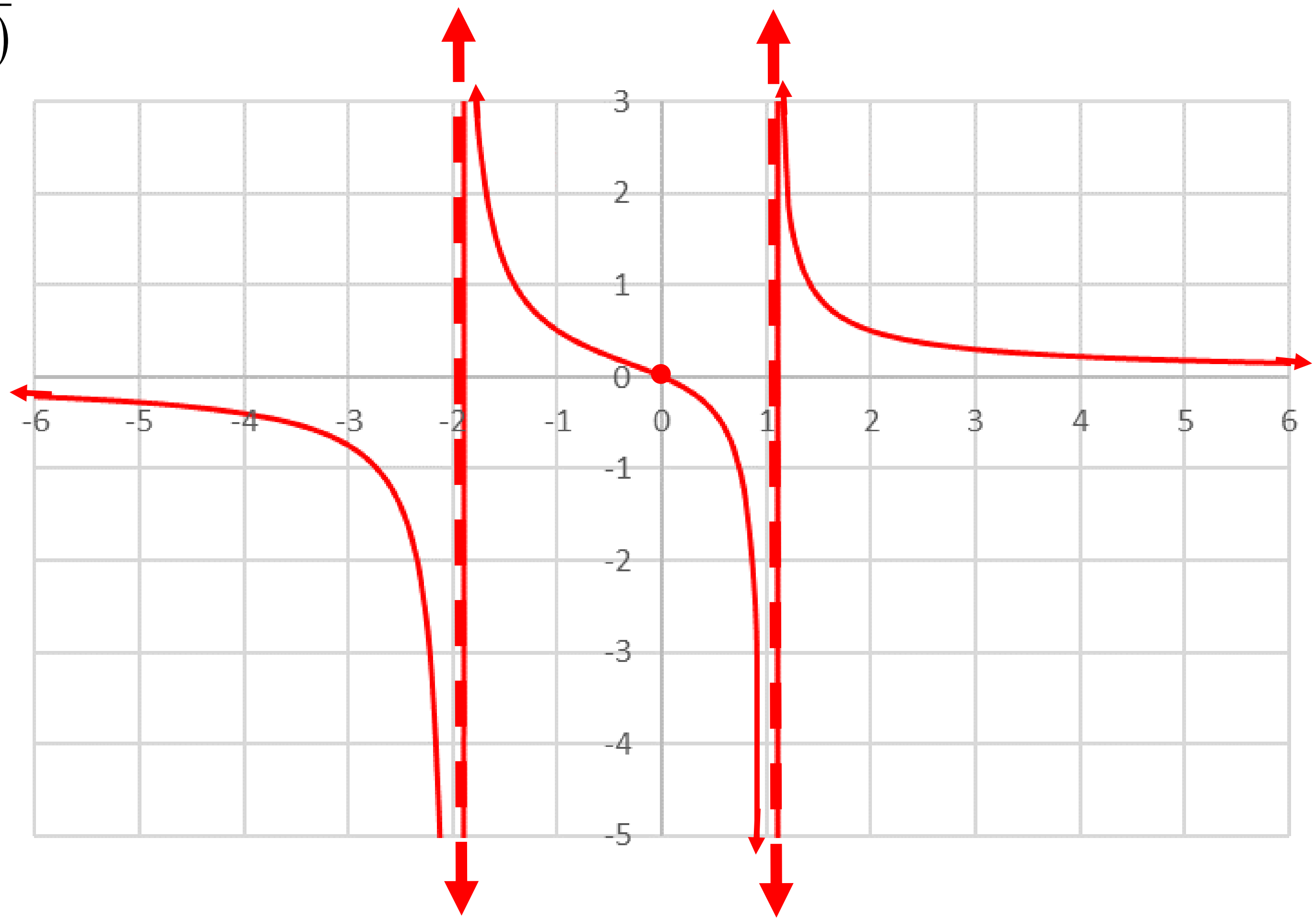
1. $f(x)=\dfrac{x}{(x-1)(x+2)}$

Asymptotes:

H.A.: $y=0$

V.A.: $x=1, x=-2$

S.A.: none



Intercepts:

***x*-int(s):** 0

***y*-int:** 0

Sign Chart:

$-$ U $+$ 0 $-$ U $+$

-2 0 1

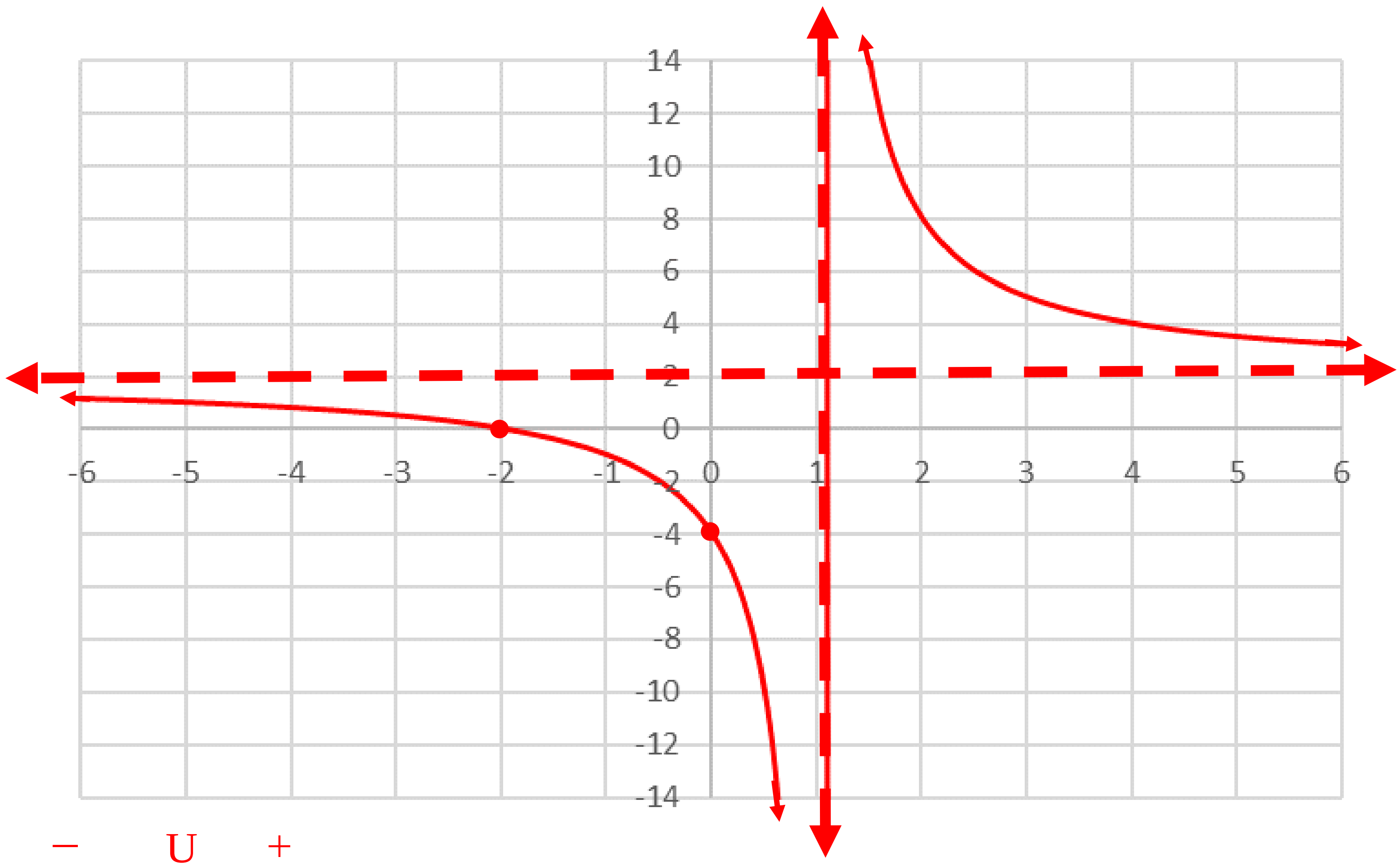
2. $f(x)=\dfrac{2x+4}{x-1}=\dfrac{2(x+2)}{x-1}$

Asymptotes:

H.A.: $y=2$

V.A.: $x=1$

S.A.: none

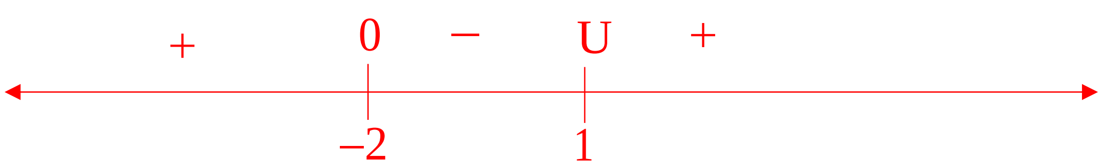


Intercepts:

***x*-int(s):** -2

***y*-int:** -4

Sign Chart:



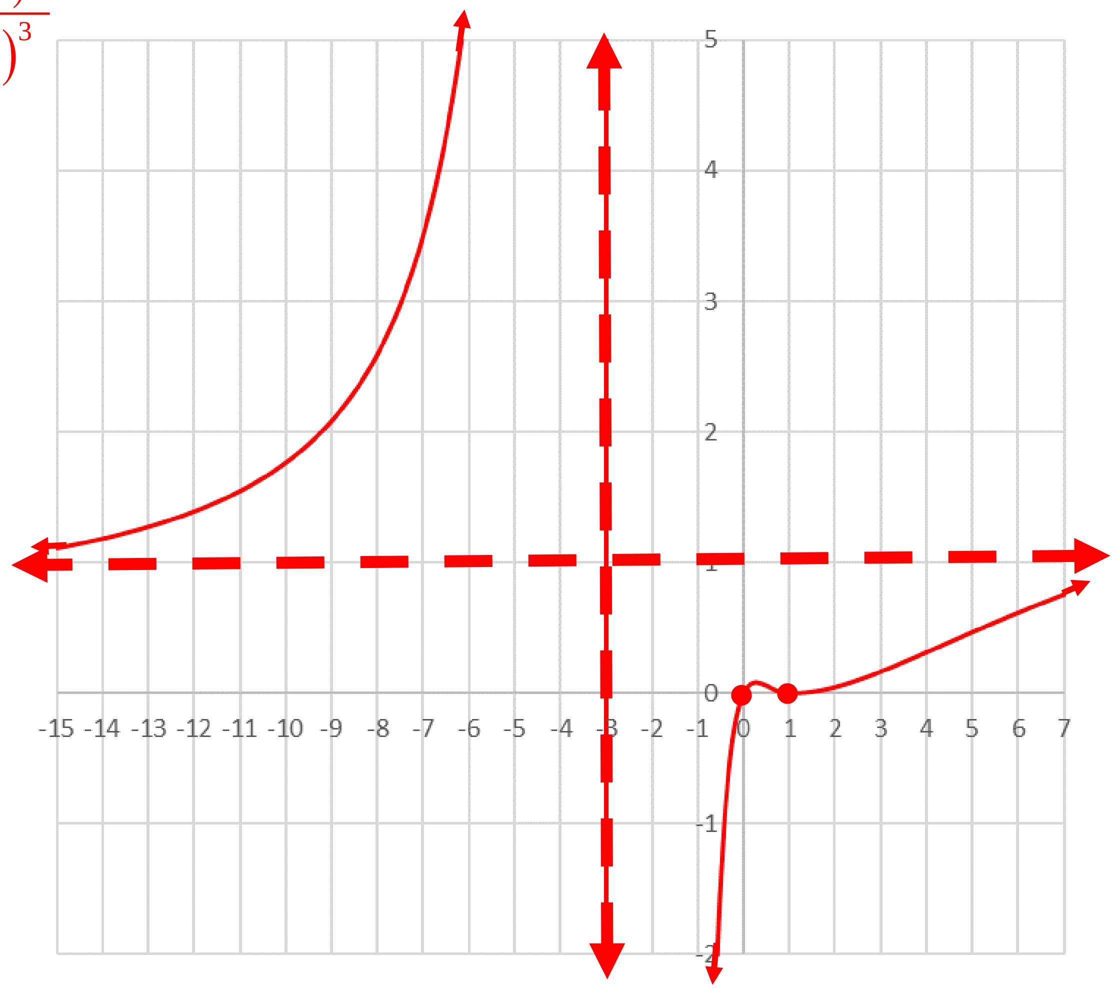
3. $f(x)=\dfrac{x(x-1)^2(x^2+1)}{(x+3)^3(x^2+1)}=\dfrac{x(x-1)^2}{(x+3)^3}$

Asymptotes:

H.A.: $y=1$

V.A.: $x=-3$

S.A.: none

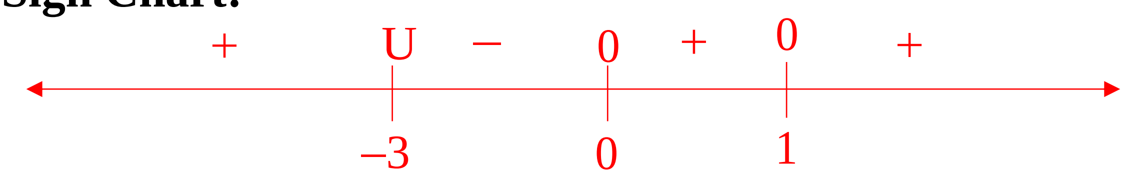


Intercepts:

***x*-int(s):** 0,1

***y*-int:** 0

Sign Chart:



4. $f(x) = \dfrac{x^2+5x+6}{x-3} = \dfrac{(x+2)(x+3)}{x-3}$

Asymptotes:

H.A.: none

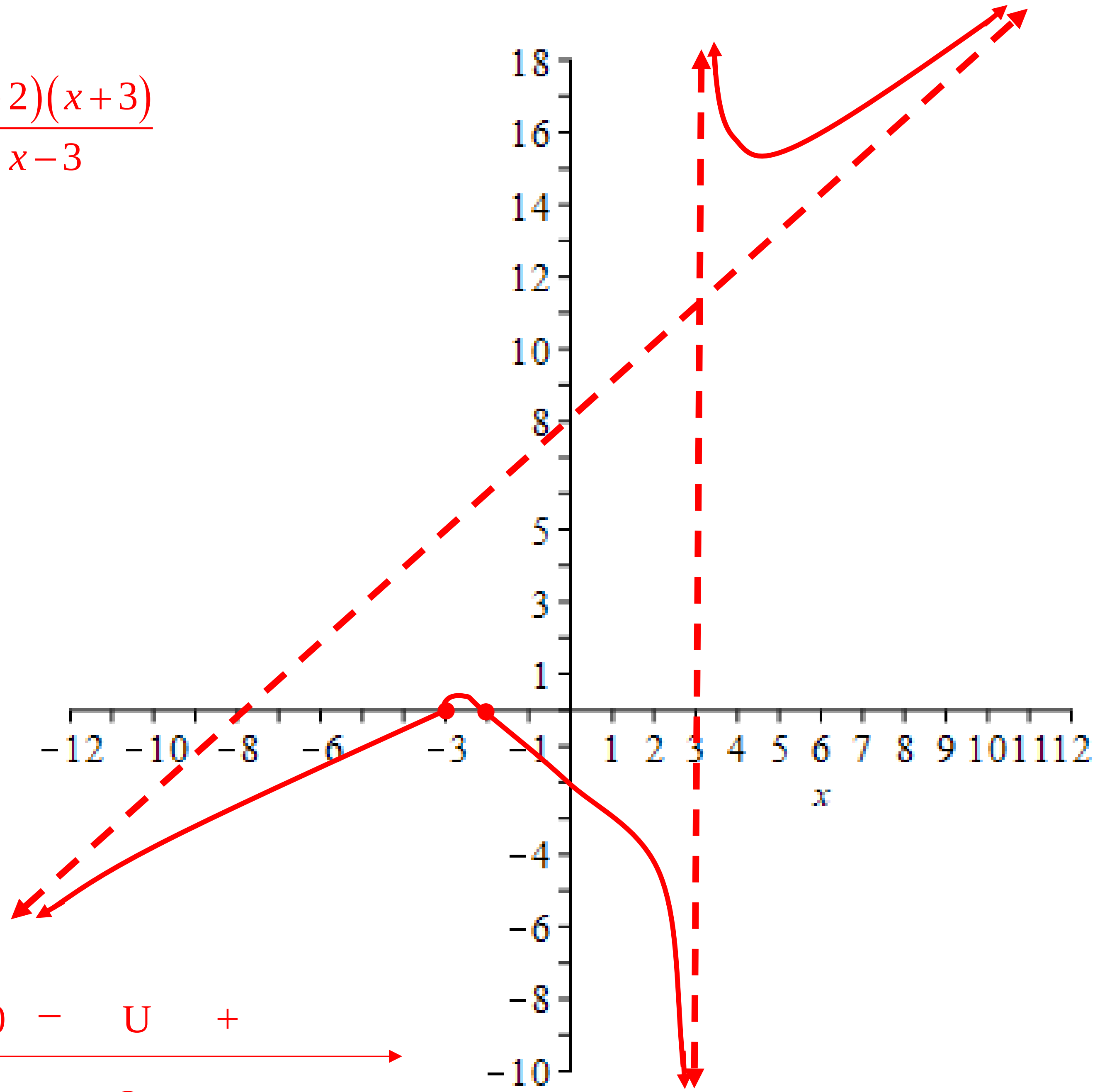
V.A.: $x = 3$

S.A.: $y = x+8$

Intercepts:

***x*-int(s):** $-2, -3$

***y*-int:** -2

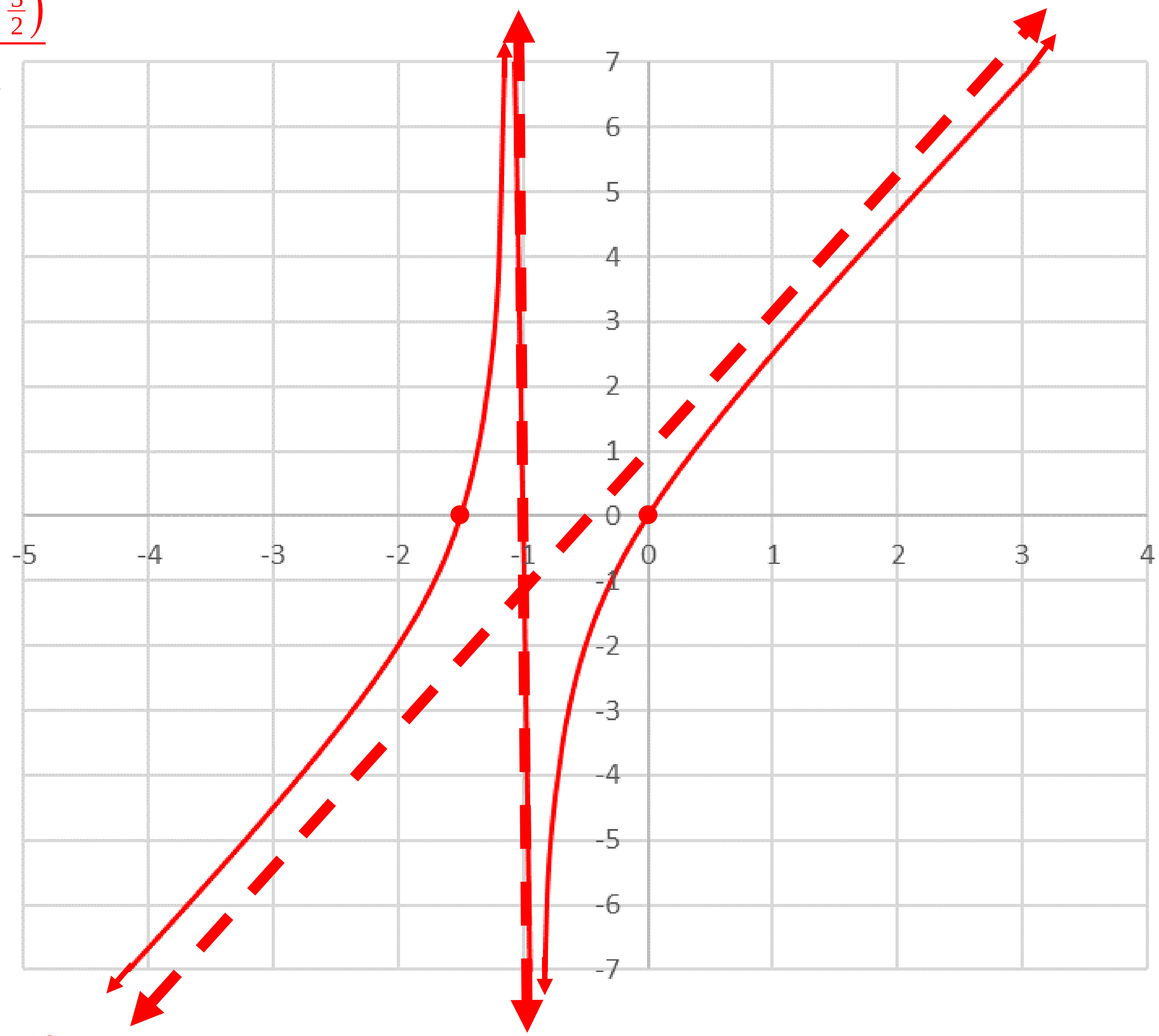


Sign Chart:

$-$ 0 $+$ 0 $-$ U $+$

-3 -2 3

5. $f(x)=\dfrac{2x^2+3x}{x+1}=\dfrac{2x\left(x+\frac{3}{2}\right)}{x+1}$



Asymptotes:

H.A.: none

V.A.: $x=-1$

S.A.: $y=2x+1$

Intercepts:

***x*-int(s):** $0, -\frac{3}{2}$

***y*-int:** 0

Sign Chart:

$-$ 0 $+$ U $-$ 0 $+$

$-\frac{3}{2}$ -1 0

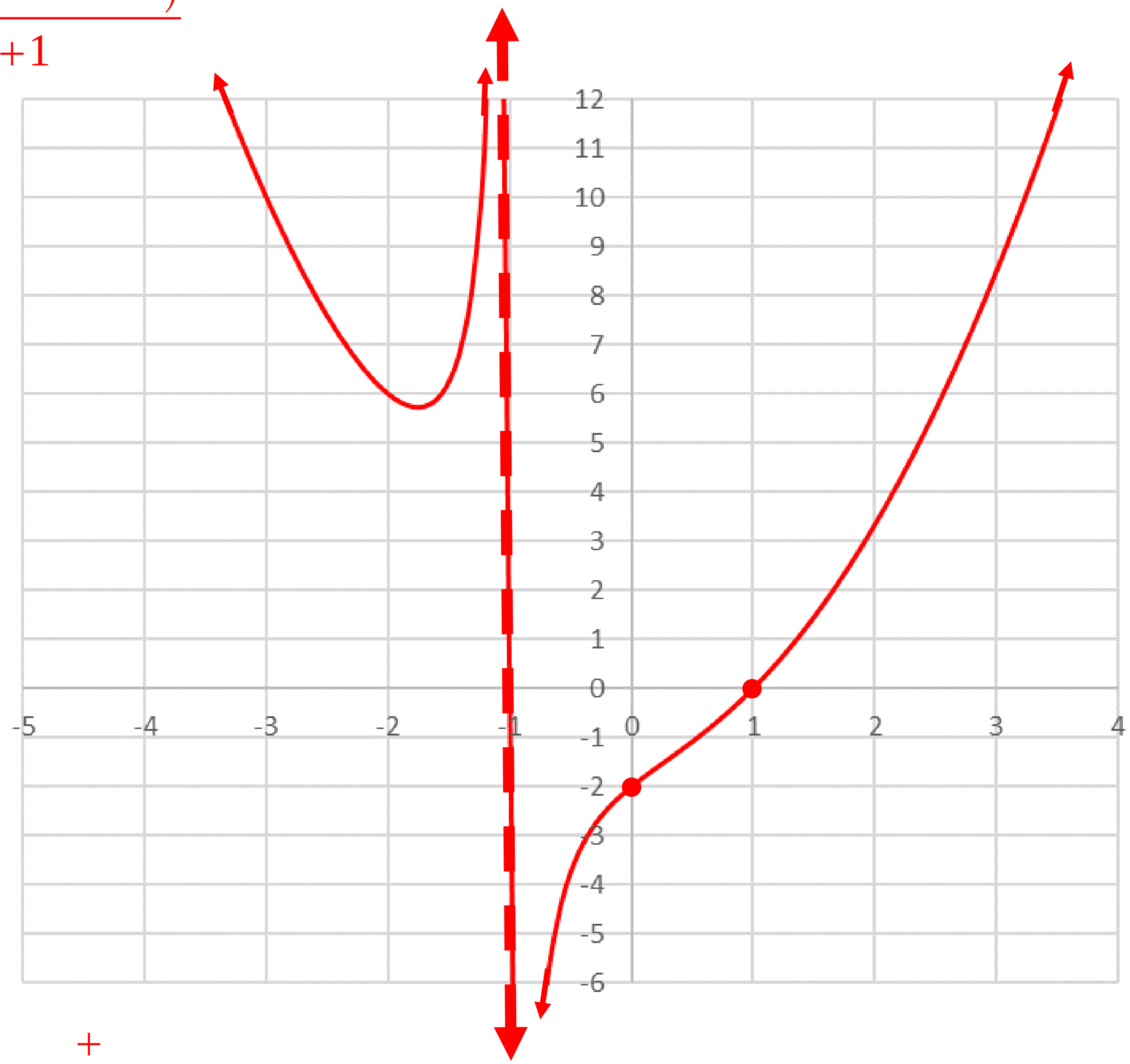
6. $f(x)=\dfrac{x^3+x^2-2}{x+1}=\dfrac{(x-1)(x^2+2x+2)}{x+1}$

Asymptotes:

H.A.: none, end-behavior of x^2

V.A.: $x=-1$

S.A.: none



Intercepts:

***x*-int(s):** 1

***y*-int:** -2

Sign Chart:

$+ \quad 0 \quad - \quad 0 \quad +$

$-1 \qquad 1$

<u>Unreduced Rational Functions:</u>

$f(x)=\dfrac{p(x)}{q(x)}$**, where** $p(x)$ **and** $q(x)$ ***<u>have</u>*** **common real zeros.**

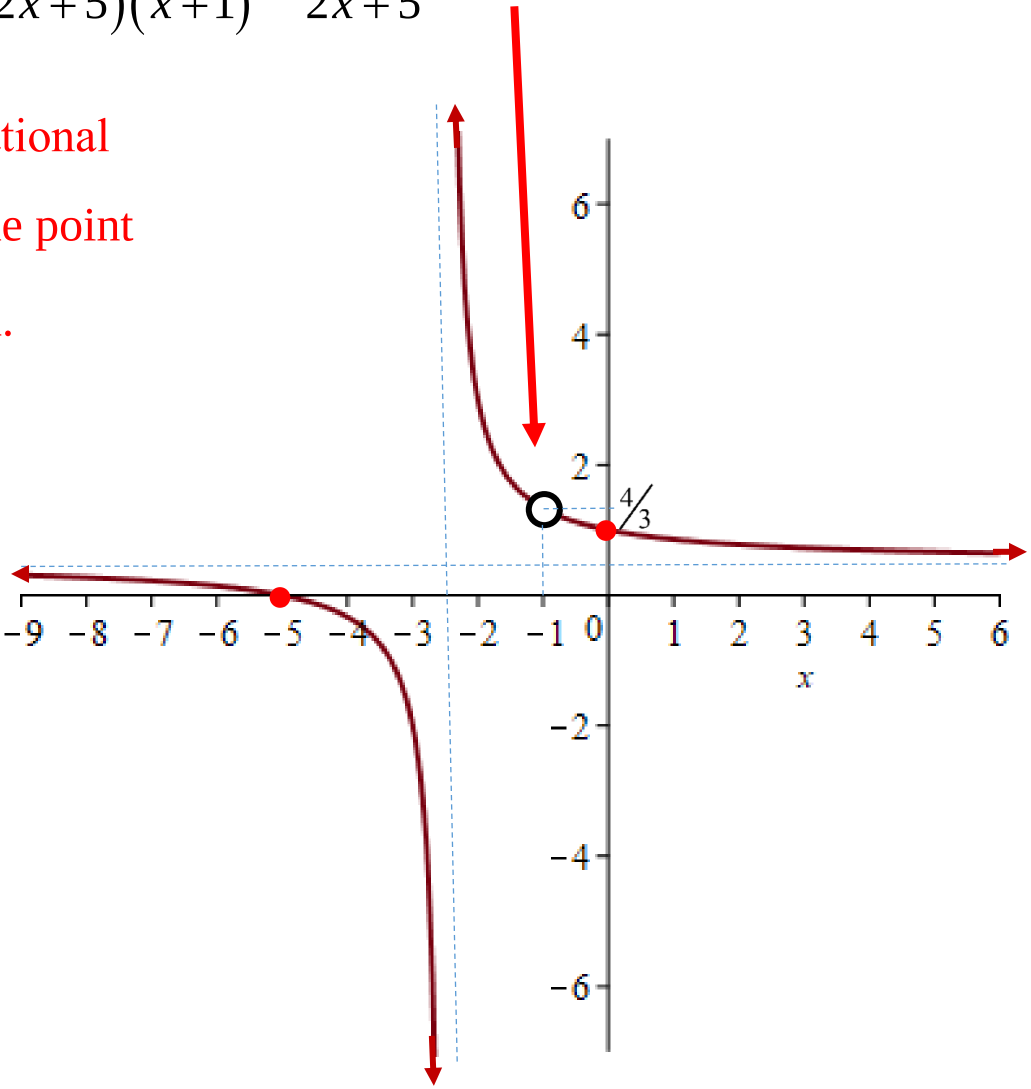
The real zeros of $q(x)$ **are numbers that are not in the domain of** $f(x)$**.**

The first step is to cancel the common factors corresponding to the common real zeros, and analyze the remaining reduced rational function with a restricted domain. ***Again, if they have common imaginary zeros, then the corresponding common factors should be cancelled out, as well.***

Examples:

1. $f(x)=\dfrac{x^2+6x+5}{2x^2+7x+5}=\dfrac{(x+1)(x+5)}{(2x+5)(x+1)}=\dfrac{x+5}{2x+5};x\neq-1$

It's the graph of the reduced rational function $g(x)=\dfrac{x+5}{2x+5}$ with the point with *x*-coordinate, -1, removed.



2. $f(x)=\dfrac{x^2+x-30}{x+6}=\dfrac{(x+6)(x-5)}{x+6}=x-5; x\neq -6$

It's the graph of the linear function $g(x)=x-5$ with the point with *x*-coordinate, -6, removed.