Review of Polynomial and Rational Inequalities:

To solve a polynomial or rational inequality, just do the following steps:

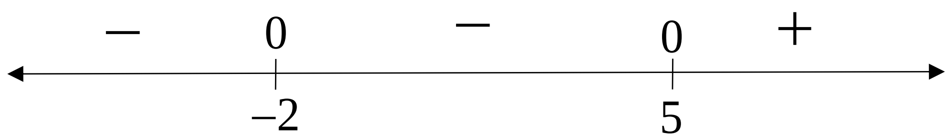
1. Get zero on one side.

2. Create the sign chart for the other side. ***(If factors can be cancelled out, then do so.)***

3. Read the solution from the sign chart.

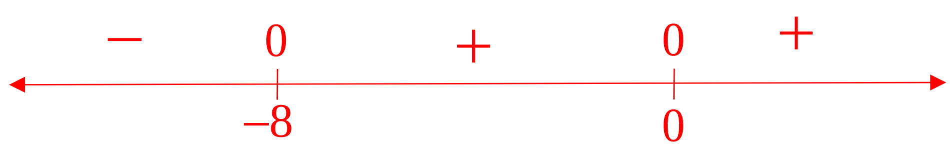
Examples:

1. $(x-5)(x+2)^2 > 0$



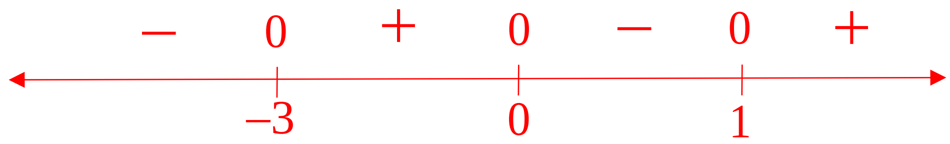
The solution in interval notation is $(5,\infty)$**.**

2. $x^3 + 8x^2 < 0 \Rightarrow x^2(x+8) < 0$



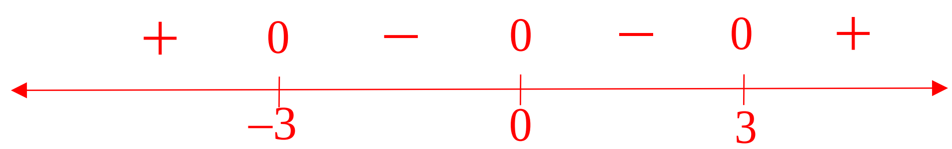
$(-\infty,-8)$

3. $x^3 + 2x^2 - 3x \geq 0 \Rightarrow x(x+3)(x-1) \geq 0$



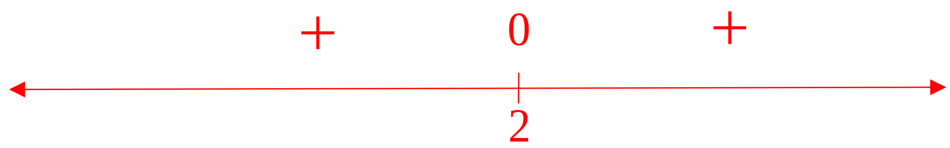
$$[-3,0] \cup [1,\infty)$$

4. $x^4 \leq 9x^2 \Rightarrow x^4 - 9x^2 \leq 0 \Rightarrow x^2(x-3)(x+3) \leq 0$



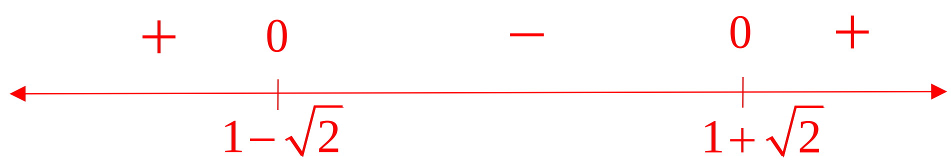
$$[-3,3]$$

5. $x^2 + 4 \le 4x \Rightarrow x^2 - 4x + 4 \le 0 \Rightarrow (x-2)^2 \le 0$



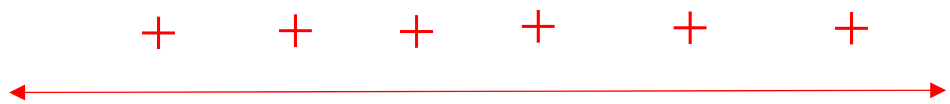
$\{2\}$

6. $x^2 - 4x \le -2 \Rightarrow x^2 - 4x + 2 \le 0$



$\left[1-\sqrt{2}, 1+\sqrt{2}\right]$

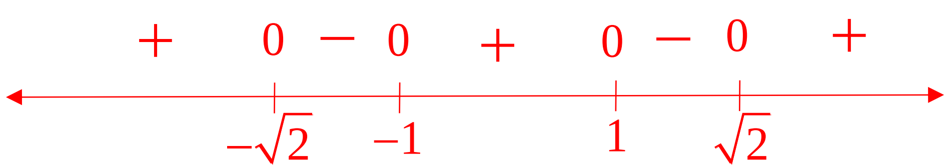
7. $\underbrace{x^2 - 4x + 5}_{\text{no real zeros}} < 0$



No solution

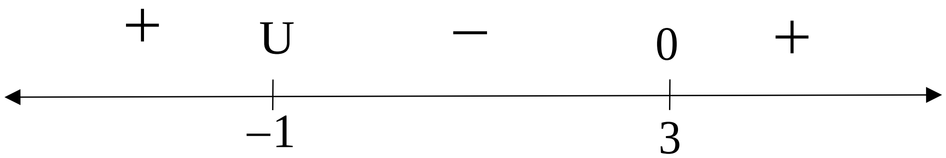
8. What's the domain of the function $f(x) = \sqrt{x^4 - 3x^2 + 2}$**?**

$$x^4 - 3x^2 + 2 \geq 0 \Rightarrow (x^2 - 1)(x^2 - 2) \geq 0 \Rightarrow (x-1)(x+1)\left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right) \geq 0$$



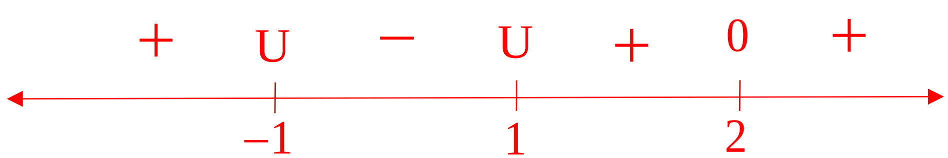
$$\left(-\infty, -\sqrt{2}\right] \cup [-1, 1] \cup \left[\sqrt{2}, \infty\right)$$

9. $\dfrac{x-3}{x+1} \geq 0$



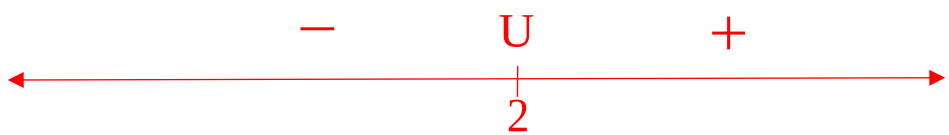
The solution in interval notation is $(-\infty,-1)\cup[3,\infty)$**.**

10. $\dfrac{(x-2)^2}{x^2-1} \geq 0 \Rightarrow \dfrac{(x-2)^2}{(x-1)(x+1)} \geq 0$



$(-\infty,-1)\cup(1,\infty)$

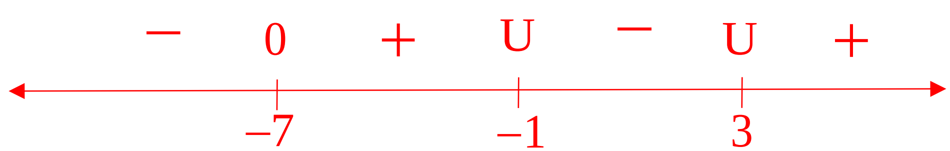
11. $\dfrac{x+4}{x-2} \le 1 \Rightarrow \dfrac{x+4}{x-2} - \dfrac{x-2}{x-2} \le 0 \Rightarrow \dfrac{6}{x-2} \le 0$



$(-\infty, 2)$

12. $\dfrac{5}{x-3} > \dfrac{3}{x+1} \Rightarrow \dfrac{5}{x-3} - \dfrac{3}{x+1} > 0 \Rightarrow \dfrac{5(x+1)}{(x-3)(x+1)} - \dfrac{3(x-3)}{(x+1)(x-3)} > 0$

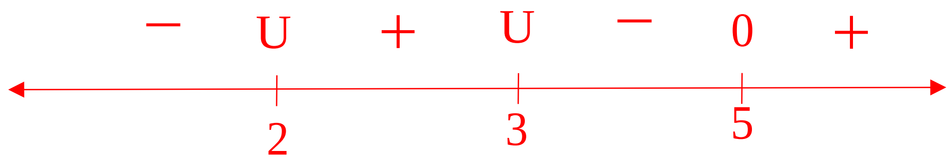
$\Rightarrow \dfrac{2(x+7)}{(x-3)(x+1)} > 0$



$(-7,-1) \cup (3, \infty)$

13. $\frac{1}{x-2} < \frac{2}{3x-9} \Rightarrow \frac{1}{x-2} - \frac{2}{3x-9} < 0 \Rightarrow \frac{3x-9}{(x-2)(3x-9)} - \frac{2(x-2)}{(x-2)(3x-9)} < 0$

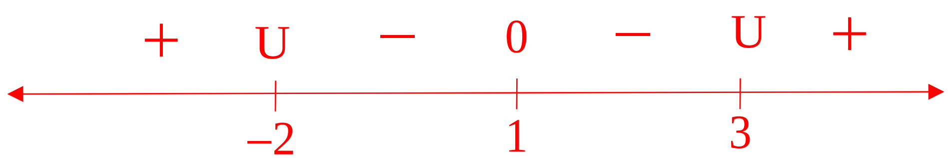
$$\Rightarrow \frac{x-5}{3(x-2)(x-3)} < 0$$



$$(-\infty,2) \cup (3,5)$$

14. What's the domain of the function $f(x) = \sqrt{\frac{(x-1)^2}{(x+2)(x-3)}}$ **?**

$$\frac{(x-1)^2}{(x+2)(x-3)} \geq 0$$



$$(-\infty,-2) \cup \{1\} \cup (3,\infty)$$

Partial Fraction Decompositions:

Algebraic fractions can be added by finding a common denominator, modifying the original numerators, and then combining them.

$$\begin{aligned}\frac{1}{x-1}+\frac{2}{x+2} &= \frac{x+2}{(x-1)(x+2)}+\frac{2(x-1)}{(x-1)(x+2)}\\ &= \frac{x+2}{(x-1)(x+2)}+\frac{2x-2}{(x-1)(x+2)}\\ &= \frac{3x}{(x-1)(x+2)}\end{aligned}$$

A partial fraction decomposition is a reversal of this process.

Start with an algebraic fraction like $\frac{3x}{(x-1)(x+2)}$**, and determine the algebraic fractions whose sum would be** $\frac{3x}{(x-1)(x+2)}$**. In this case it's** $\frac{1}{x-1}+\frac{2}{x+2}$**.**

In general, you'll start with a rational function, $\frac{p(x)}{q(x)}$, and you'll try to write it as

$$\frac{p(x)}{q(x)} = \underbrace{f_1(x) + f_2(x) + \cdots + f_r(x)}_{\text{Partial Fraction Decomposition}}.$$

The process begins by forming a guess for the decomposition that contains parameters(variables) whose values are to be determined.

The values of the parameters can be determined by plugging in values of *x*, matching up coefficients, or a combination of the two.

When the values of the parameters are plugged into the guess, the partial fraction decomposition will be complete.

Assuming that the degree of $q(x)$ is greater than the degree of $p(x)$,

1. Factor $q(x)$.

2. If $(ax+b)^k$ is the highest power of $(ax+b)$ which is a factor of , $q(x)$ then your guess must include the terms $\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k}$ **.**

3. If $(ax^2+bx+c)^k$ is the highest power of (ax^2+bx+c) which is a factor of , $q(x)$ then your guess must include the terms

$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$ **, where ax^2+bx+c is an irreducible quadratic factor, i.e. it doesn't have real zeros.**

Examples:

1. $\dfrac{3x}{(x+2)(x-4)}$

Formulate the guess.

$\dfrac{3x}{(x+2)(x-4)} = \dfrac{A}{x+2} + \dfrac{B}{x-4}$**, so multiply both sides by the denominator on the left to get**

$$3x = A(x-4) + B(x+2)$$

Method#1: Expand the right side, collect like terms, and equate coefficients and constants on both sides.

$3x = Ax - 4A + Bx + 2B \Rightarrow 3x + 0 = (A+B)x + (2B-4A)$**, so** $\begin{matrix} A+B=3 \\ 2B-4A=0 \end{matrix}$**.**

Solving the system leads to $A=1, B=2$**.**

Method#2: Pick numerical values for *x* that make it easy to determine the values of *A* and *B*.

Let $x = 4$ **to get the equation** $12 = 6B \Rightarrow B = 2$**, and let** $x = -2$ **to get the equation** $-6 = -6A \Rightarrow A = 1$**.**

Either method leads to a partial fraction decomposition of

$$\frac{3x}{(x+2)(x-4)} = \frac{1}{x+2} + \frac{2}{x-4}.$$

2. $\dfrac{x}{x^2+2x-3}$ $\qquad \{x^2+2x-3=(x+3)(x-1)\}$

Formulate the guess and multiply by the denominator on the left.

$$\frac{x}{(x+3)(x-1)}=\frac{A}{x+3}+\frac{B}{x-1}\Rightarrow x=A(x-1)+B(x+3)$$

Let $x=1$ to get $1=4B\Rightarrow B=\dfrac{1}{4}$.

Let $x=-3$ to get $-3=-4A\Rightarrow A=\dfrac{3}{4}$.

So $\dfrac{x}{x^2+2x-3}=\dfrac{3}{4(x+3)}+\dfrac{1}{4(x-1)}$.

3. $\dfrac{1}{x(x^2+1)}$

Formulate the guess and multiply by the denominator on the left.

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} \Rightarrow 1 = A(x^2+1) + x(Bx+C)$$

$$\Rightarrow 1 = (A+B)x^2 + Cx + A$$

Expand the right side and equate coefficients and constant.

So $A=1, C=0, A+B=0 \Rightarrow B=-1.$

So $\dfrac{1}{x(x^2+1)} = \dfrac{1}{x} - \dfrac{x}{x^2+1}.$

4. $\dfrac{x+1}{x^2(x-2)}$

Formulate the guess and multiply by the denominator on the left.

$$\frac{x+1}{x^2(x-2)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-2}\Rightarrow x+1=Ax(x-2)+B(x-2)+Cx^2$$

Let $x=0$ to get $1=-2B\Rightarrow B=-\dfrac{1}{2}$.

Let $x=2$ to get $3=4C\Rightarrow C=\dfrac{3}{4}$.

Let $x=1$ to get $2=-A-B+C\Rightarrow 2=-A+\dfrac{1}{2}+\dfrac{3}{4}\Rightarrow A=-\dfrac{3}{4}$.

So $\dfrac{x+1}{x^2(x-2)}=-\dfrac{3}{4x}-\dfrac{1}{2x^2}+\dfrac{3}{4(x-2)}$.

5. $\dfrac{x+4}{x^2\left(x^2+4\right)}$

Formulate the guess and multiply by the denominator on the left.

$$\frac{x+4}{x^2\left(x^2+4\right)}=\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+4}\Rightarrow x+4=Ax\left(x^2+4\right)+B\left(x^2+4\right)+x^2\left(Cx+D\right)$$

$$=Ax^3+4Ax+Bx^2+4B+Cx^3+Dx^2$$

$$=\left(A+C\right)x^3+\left(B+D\right)x^2+4Ax+4B$$

Expand the right side and equate coefficients and constant.

So $4B=4, 4A=1, B+D=0, A+C=0 \Rightarrow B=1, A=\dfrac{1}{4}, C=-\dfrac{1}{4}, D=-1.$

$$\text{So } \frac{x+4}{x^2\left(x^2+4\right)}=\frac{1}{4x}+\frac{1}{x^2}-\frac{x+4}{4\left(x^2+4\right)}.$$

Formulate the guess and multiply by the denominator on the left.

6. $\dfrac{x^2+2x+3}{\left(x^2+4\right)^2}$

$$\frac{x^2+2x+3}{\left(x^2+4\right)^2}=\frac{Ax+B}{x^2+4}+\frac{Cx+D}{\left(x^2+4\right)^2}\Rightarrow x^2+2x+3=\left(Ax+B\right)\left(x^2+4\right)+Cx+D$$

$$=Ax^3+4Ax+Bx^2+4B+Cx+D$$

$$=Ax^3+Bx^2+\left(4A+C\right)x+\left(4B+D\right)$$

Expand the right side and equate coefficients and constant.

So $A=0, B=1, 4A+C=2, 4B+D=3\Rightarrow C=2, D=-1.$

So $\dfrac{x^2+2x+3}{\left(x^2+4\right)^2}=\dfrac{1}{x^2+4}+\dfrac{2x-1}{\left(x^2+4\right)^2}$.

7. $\dfrac{x^3+x^2-3}{x^2+3x-4}$ **{Divide first.}**

$$
\begin{array}{r}
x-2 \\
x^2+3x-4\;\big)\;\overline{x^3+x^2-3} \\
-\left(x^3+3x^2-4x\right) \\
\hline
-2x^2+4x-3 \\
-\left(-2x^2-6x+8\right) \\
\hline
10x-11
\end{array}
$$

$$\frac{x^3+x^2-3}{x^2+3x-4}=x-2+\frac{10x-11}{x^2+3x-4}=x-2+\frac{10x-11}{(x+4)(x-1)}=x-2+\frac{51}{5(x+4)}-\frac{1}{5(x-1)}$$

8. Find the exact value of the following sum of 999,999 terms:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{999{,}999\cdot 1{,}000{,}000}$$

A general term in the sum looks like $\frac{1}{x(x+1)}$, so let's do a partial fraction decomposition of it.

$$\frac{1}{x(x+1)}=\frac{A}{x}+\frac{B}{x+1}\Rightarrow 1=A(x+1)+Bx\Rightarrow\frac{1}{x(x+1)}=\frac{1}{x}-\frac{1}{x+1}$$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{999{,}999\cdot 1{,}000{,}000}$$

$$=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{999{,}999}-\frac{1}{1{,}000{,}000}\right)$$

The middle terms between 1 and $-\frac{1}{1{,}000{,}000}$ cancel out.

$$=1-\frac{1}{1{,}000{,}000}=\frac{999{,}999}{1{,}000{,}000}$$

9. Find the exact value of the following sum of 999,998 terms:

$$\frac{2}{1\cdot 2\cdot 3}+\frac{2}{2\cdot 3\cdot 4}+\frac{2}{3\cdot 4\cdot 5}+\cdots+\frac{2}{999{,}998\cdot 999{,}999\cdot 1{,}000{,}000}$$

A general term in the sum looks like $\frac{2}{x(x+1)(x+2)}$**, and if you do a partial fraction decomposition, you'll get**

$$\frac{2}{x(x+1)(x+2)}=\frac{1}{x}-\frac{2}{x+1}+\frac{1}{x+2}=\left(\frac{1}{x}-\frac{1}{x+1}\right)-\left(\frac{1}{x+1}-\frac{1}{x+2}\right).$$

So

$$\frac{2}{1\cdot 2\cdot 3}+\frac{2}{2\cdot 3\cdot 4}+\frac{2}{3\cdot 4\cdot 5}+\cdots+\frac{2}{999{,}998\cdot 999{,}999\cdot 1{,}000{,}000}$$

$$=\left[\left(1-\tfrac{1}{2}\right)+\left(\tfrac{1}{2}-\tfrac{1}{3}\right)+\left(\tfrac{1}{3}-\tfrac{1}{4}\right)+\cdots+\left(\tfrac{1}{999{,}998}-\tfrac{1}{999{,}999}\right)\right]-\left[\left(\tfrac{1}{2}-\tfrac{1}{3}\right)+\left(\tfrac{1}{3}-\tfrac{1}{4}\right)+\left(\tfrac{1}{4}-\tfrac{1}{5}\right)+\cdots+\left(\tfrac{1}{999{,}999}-\tfrac{1}{1{,}000{,}000}\right)\right]$$

The middle terms between 1 and $-\tfrac{1}{999{,}999}$ cancel, and the middle terms between $\tfrac{1}{2}$ and $-\tfrac{1}{1{,}000{,}000}$ cancel.

$$=1-\tfrac{1}{999{,}999}-\tfrac{1}{2}+\tfrac{1}{1{,}000{,}000}=\tfrac{1}{2}-\tfrac{1}{999{,}999}+\tfrac{1}{1{,}000{,}000}$$