

Review of Properties of Logarithms:

For M and N positive numbers and r a real number,

Product Rule:

Expansion



$$\log_b(MN) = \log_b M + \log_b N$$



Compression

Why? $b^{\log_b(MN)} = MN$

and $b^{(\log_b M + \log_b N)} = b^{\log_b M} \cdot b^{\log_b N} = MN$

Expand and simplify:

$$\log_5(25x)$$


$$\log_5(25x) = \log_5 25 + \log_5 x = \boxed{2 + \log_5 x}$$


Compress and simplify:

$$\log_6 9 + \log_6 4$$

$$\log_6 9 + \log_6 4 = \log_6 36 = \boxed{2}$$

Quotient Rule:

Expansion

$$\log_b \left(\frac{M}{N} \right) = \log_b M - \log_b N$$


Compression

Why? $b^{\log_b \left(\frac{M}{N} \right)} = \frac{M}{N}$ and $b^{(\log_b M - \log_b N)} = \frac{b^{\log_b M}}{b^{\log_b N}} = \frac{M}{N}$

Expand and simplify:

$$\log_3 \left(\frac{x}{9} \right)$$

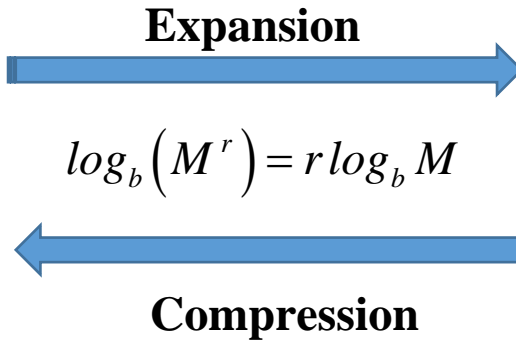
$$\log_3 \left(\frac{x}{9} \right) = \log_3 x - \log_3 9 = \boxed{\log_3 x - 2}$$

Compress and simplify:

$$\log_3 2 - \log_3 6$$

$$\log_3 2 - \log_3 6 = \log_3 \left(\frac{1}{3} \right) = \boxed{-1}$$

Power Rule:



Why? $b^{\log_b(M^r)} = \mathbf{M^r}$

and $b^{r \log_b M} = (b^{\log_b M})^r = \mathbf{M^r}$

Expand and simplify:

$$\log_7(7x^5)$$

$$\log_7(7x^5) = \log_7 7 + 5\log_7 x = \boxed{1 + 5\log_7 x}$$

Compress:

$$2\log_3 x - 4\log_3 y$$

$$2\log_3 x - 4\log_3 y = \log_3 x^2 - \log_3 y^4 = \boxed{\log_3 \left(\frac{x^2}{y^4} \right)}$$

Expand: $\log_2 \left[\frac{x^3(x+2)}{(x+3)^2} \right]$

$$\begin{aligned} \log_2 \left[\frac{x^3(x+2)}{(x+3)^2} \right] &= \log_2 x^3 + \log_2(x+2) - \log_2 \left[(x+3)^2 \right] \\ &= \boxed{3\log_2 x + \log_2(x+2) - 2\log_2(x+3)} \end{aligned}$$

Compress: $3\log_5(3x+1) - 2\log_5(2x-1) - \log_5 x$

$$\begin{aligned} 3\log_5(3x+1) - 2\log_5(2x-1) - \log_5 x &= \log_5 \left[(3x+1)^3 \right] - \log_5 \left[(2x-1)^2 \right] \\ &= \boxed{\log_5 \left(\frac{(3x+1)^3}{(2x-1)^2} \right)} \end{aligned}$$

Change of Base Formula:

Suppose that $y = \log_b x$. Then $b^y = x$ and therefore $\log_a (b^y) = \log_a x$. From the Power

Rule, you get $y \log_a b = \log_a x$, and solving for y yields $y = \frac{\log_a x}{\log_a b}$. So

$$\log_b x = \frac{\log_a x}{\log_a b}.$$

Calculators have a logarithm key for base 10, log, called the common logarithm. They also have a logarithm key for base e , ln, called the natural logarithm. $e = 2.7182818...$

$$\log_b x = \frac{\log x}{\log b}$$

Or

$$\log_b x = \frac{\ln x}{\ln b}$$

Example:

Calculate $\log_3 5$ to 3 decimal places.

$$\log_3 5 = \frac{\log 5}{\log 3}$$

Or

$$\log_3 5 = \frac{\ln 5}{\ln 3}$$

1.465

Exponential and Logarithmic Equations:

The goal in solving exponential and logarithmic equations is to remove the exponential and logarithmic parts, eventually.

$$b^x = b^y \Rightarrow x = y$$

$$\log_b x = \log_b y \Rightarrow x = y$$

$$\log_b (b^x) = x; \text{ for all } x$$

$$b^{\log_b x} = x; \text{ for } x > 0$$

Examples:

1. $\log(x+6)=1$

$$\Rightarrow 10^{\log(x+6)} = 10^1 \Rightarrow x+6=10 \Rightarrow x = \boxed{4}$$

2. $\log_4(x+2)=\log_4 8$

$$\Rightarrow x+2=8 \Rightarrow x = \boxed{6}$$

3. $\log_4(x+2)=\log_4(2x+7)$

{Be careful!}

$$\Rightarrow x+2=2x+7 \Rightarrow x=-5$$

$$\Rightarrow \boxed{\text{no solution}}$$

-5 doesn't satisfy the original equation because it creates logarithms of negative numbers.

4. $2\log_5 x = 3\log_5 4$

$$\Rightarrow \log_5 x^2 = \log_5 4^3 \Rightarrow x^2 = 64$$

$$\Rightarrow x = \pm 8 \Rightarrow x = \boxed{8}, \text{ } -8 \text{ creates a logarithm of a negative number in the original equation.}$$

5. $\log_6(x+4) + \log_6(x+3) = 1$

$$\Rightarrow \log_6[(x+4)(x+3)] = 1 \Rightarrow (x+4)(x+3) = 6$$

$$\Rightarrow x^2 + 7x + 12 = 6 \Rightarrow x^2 + 7x + 6 = 0$$

$$\Rightarrow (x+1)(x+6) = 0 \Rightarrow x = -1, -6 \Rightarrow x = \boxed{-1}$$

-6 creates logarithms of negative numbers in the original equation.

6. $\log_3 x - 2\log_3 5 = \log_3(x+1) - 2\log_3 10$

$$\Rightarrow \log_3\left(\frac{x}{25}\right) = \log_3\left[\frac{(x+1)}{100}\right] \Rightarrow \frac{x}{25} = \frac{(x+1)}{100} \Rightarrow 100x = 25x + 25 \Rightarrow 75x = 25 \Rightarrow x = \boxed{\frac{1}{3}}$$

7. $3^{2x} + 3^x - 2 = 0$

$$\Rightarrow (3^x + 2)(3^x - 1) = 0 \Rightarrow 3^x = -2, 3^x = 1$$

$$\Rightarrow x = \boxed{0}, 3^x \text{ only produces positive values, so it can't equal } -2.$$

8. $2^{2x} + 2^{x+2} - 12 = 0$

$$\Rightarrow 2^{2x} + 4 \cdot 2^x - 12 = 0 \Rightarrow (2^x + 6)(2^x - 2) = 0$$

$$\Rightarrow 2^x = -6, 2^x = 2 \Rightarrow x = \boxed{1}, 2^x \text{ produces only positive values, so it can't equal } -6.$$

9. $3^{1-2x} = 4^x$

$$\Rightarrow \ln(3^{1-2x}) = \ln(4^x) \Rightarrow (1-2x)\ln 3 = x\ln 4 \Rightarrow \ln 3 = (2\ln 3 + \ln 4)x$$

$$\Rightarrow x = \boxed{\frac{\ln 3}{2\ln 3 + \ln 4}}$$

$$10. 5^{2x} - 8 \cdot 5^x = -16$$

$$\Rightarrow 5^{2x} - 8 \cdot 5^x + 16 = 0 \Rightarrow (5^x - 4)^2 = 0$$

$$\Rightarrow 5^x = 4 \Rightarrow x = \boxed{\frac{\log 4}{\log 5}}$$

$$11. 3^x - 14 \cdot 3^{-x} = 5$$

$$\Rightarrow 3^x (3^x - 14 \cdot 3^{-x}) = 5 \cdot 3^x \Rightarrow 3^{2x} - 14 = 5 \cdot 3^x \Rightarrow 3^{2x} - 5 \cdot 3^x - 14 = 0$$

$$\Rightarrow (3^x - 7)(3^x + 2) = 0 \Rightarrow 3^x = 7, 3^x = -2 \Rightarrow x = \boxed{\frac{\ln 7}{\ln 3}}$$

3^x produces only positive values, so it can't equal -2 .

Approximating Solutions of Equations Using the Method of Successive Approximation:

Step #1: Arrange the equation into the form $x = f(x)$.

Step #2: Choose a starting guess/approximation for a solution, x_1 .

Step #3: Evaluate the function, f , at x_1 to get the second approximation, x_2 .

Step #4: Continue this process of evaluating the function, f , at the current approximation, x_{n-1} , to get the next approximation, x_n .

$$x_n = f(x_{n-1})$$

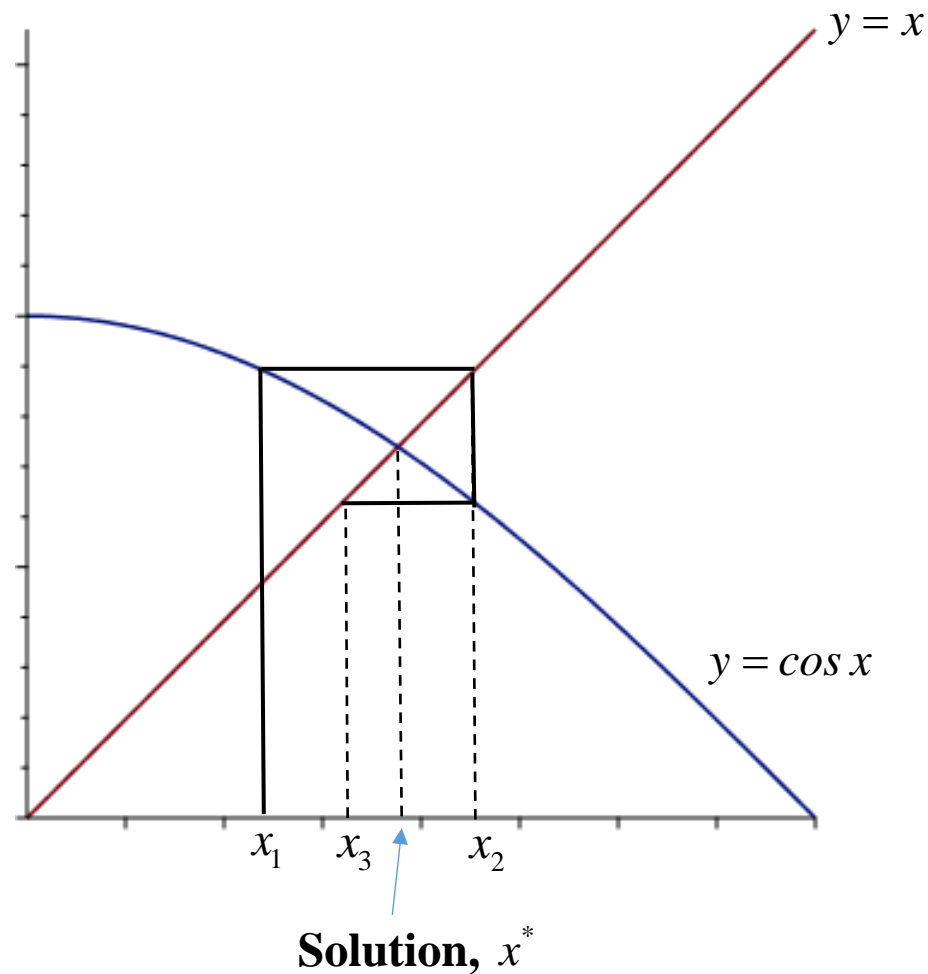
Example: Approximate a solution of the equation $x = \cos x$ using the Method of Successive Approximation with a starting guess of $x_1 = .5$.

x_1	.5
x_2	.877582561
x_3	.639012494
x_4	.8026851
x_5	.694778026
x_6	.768195831
x_7	.719165445
x_8	.752355759
x_9	.730081063
x_{10}	.745120341
x_{11}	.735006309
x_{12}	.741826522

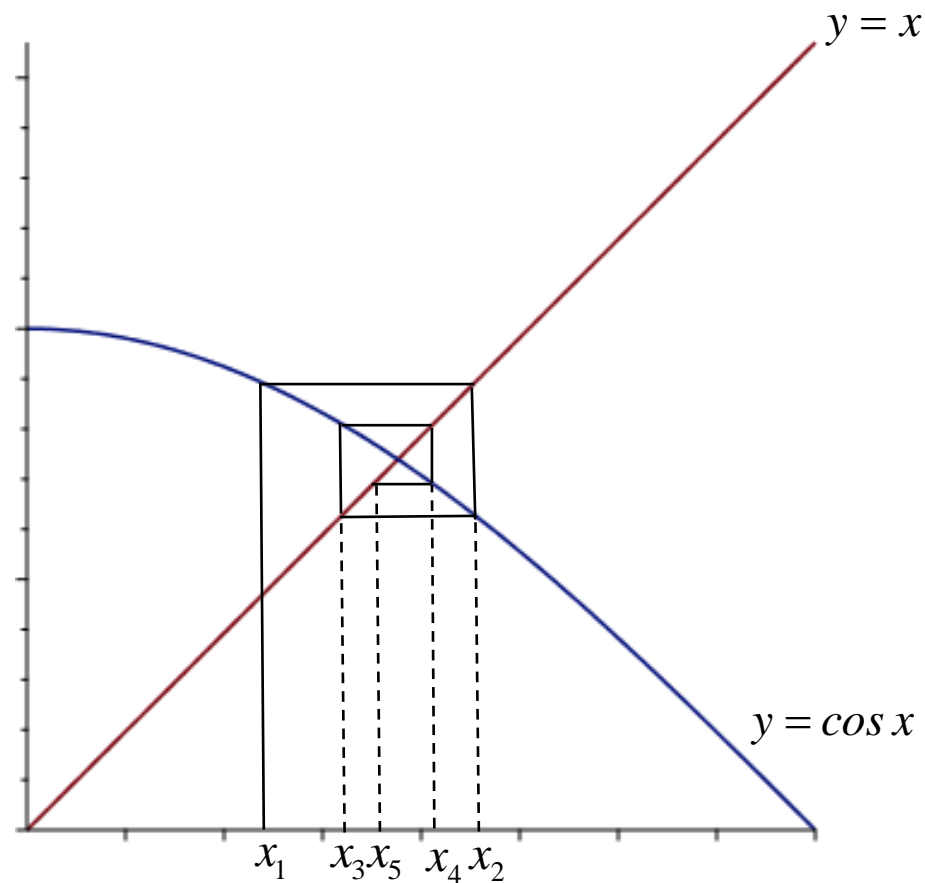
Check out the link [Successive XL](#).

How do you know if the Method of Successive Approximation is producing approximations that are getting closer to an actual solution?

Do a graphical analysis called a Cobweb Diagram. Solutions correspond to points of intersection between the graph of $y = x$ and $y = f(x)$.



**The order you follow to create the Cobweb Diagram is vertically to the curve(
 $y = f(x)$), horizontally to the line($y = x$), vertically to the curve, horizontally to the
line, ...**



Solutions for which nearby starting guesses generate approximations that get closer to the solution are called attracting solutions.

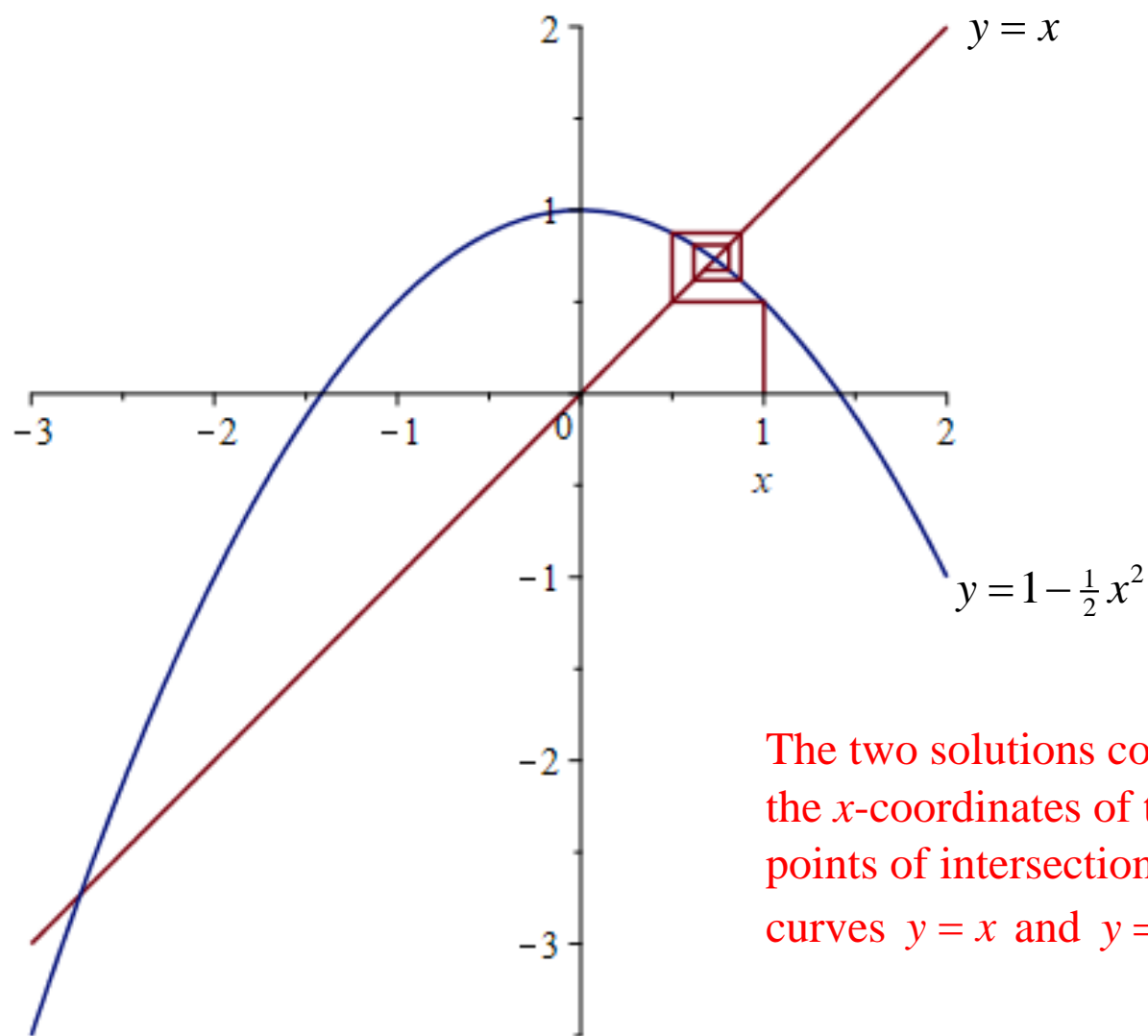
Solutions for which nearby starting guesses generate approximations that are pushed away from the solution are called repelling solutions.

Repelling solutions are invisible to this method of approximation.

The equation $1 - \frac{1}{2}x^2 = x$ has two solutions. Classify them as attracting or repelling. Demonstrate with the starting values of $x_1 = 1, 0, -2, -3$.

$$x_1 = 1$$

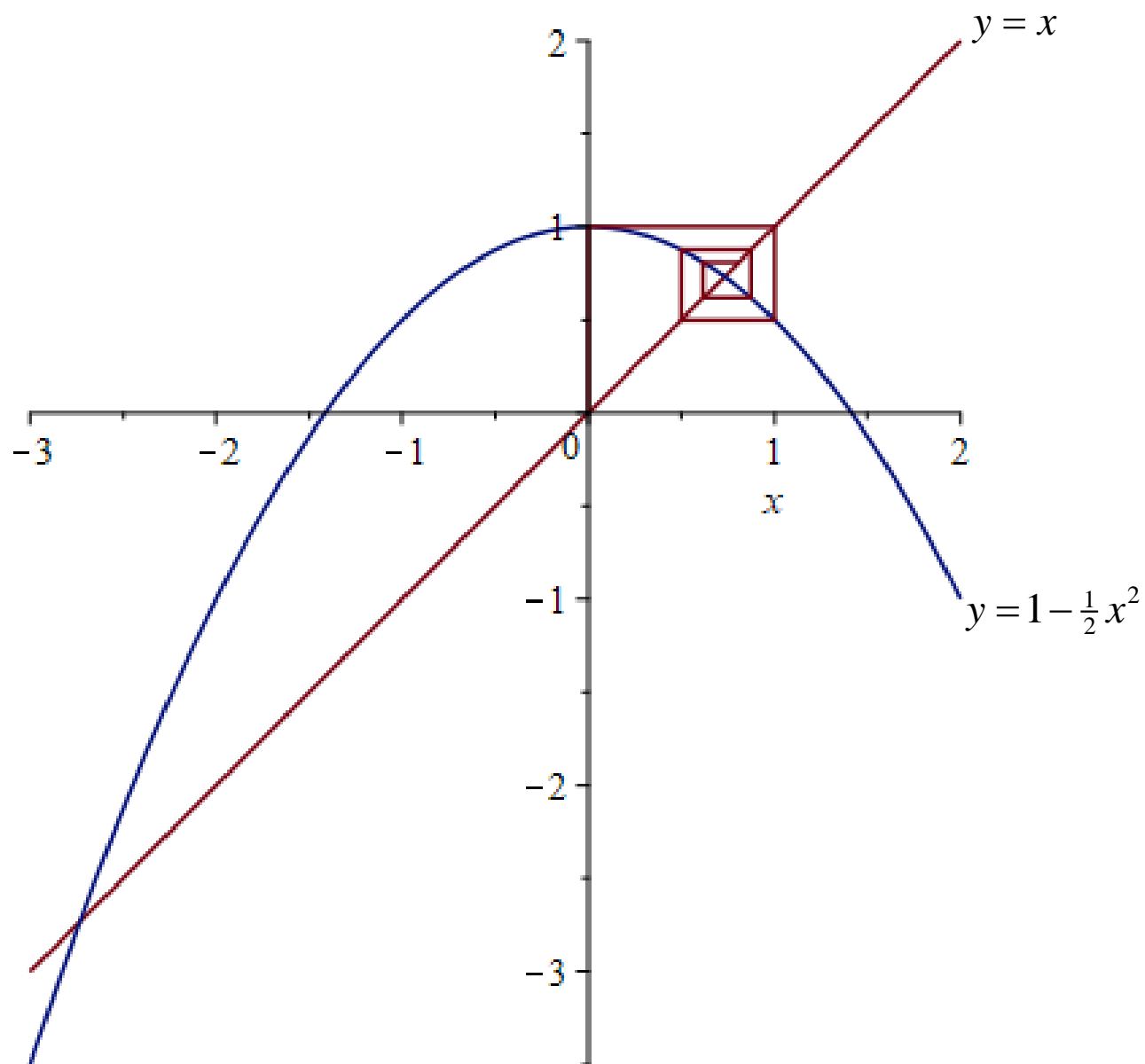
x_1	1
x_2	0.5
x_3	0.875
x_4	0.6171875
x_5	0.809539795
x_6	0.67232266
x_7	0.77399112
x_8	0.700468873
x_9	0.754671679
x_{10}	0.715235328
x_{11}	0.744219213
x_{12}	0.723068882
x_{13}	0.738585696
x_{14}	0.727245585
x_{15}	0.73555693
x_{16}	0.729478002
x_{17}	0.733930923
x_{18}	0.7306727
x_{19}	0.733058702
x_{20}	0.731312469



The two solutions correspond to the x -coordinates of the two points of intersection of the curves $y = x$ and $y = 1 - \frac{1}{2}x^2$.

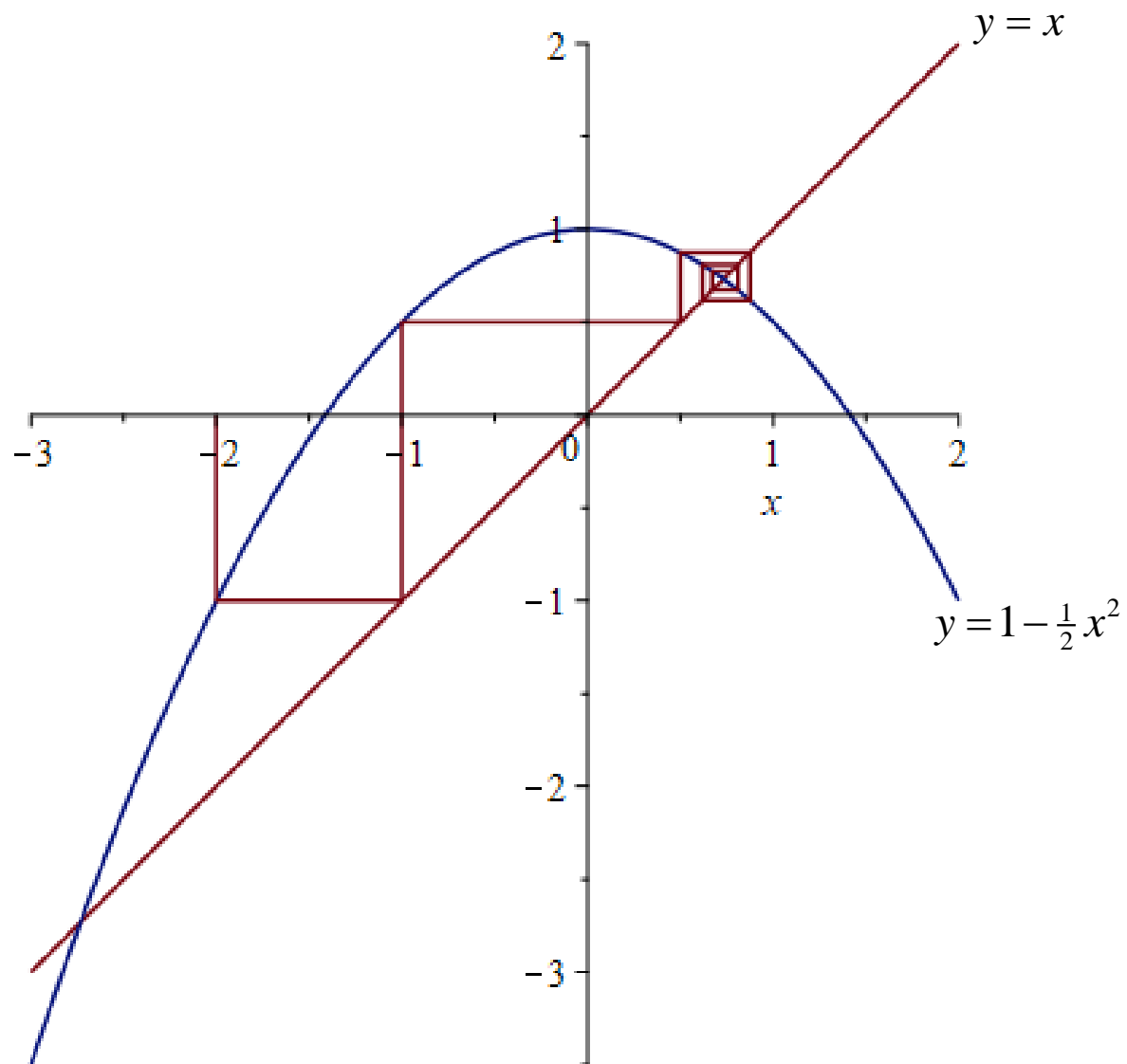
$$x_1 = 0$$

x_1	0
x_2	1
x_3	0.5
x_4	0.875
x_5	0.6171875
x_6	0.809539795
x_7	0.67232266
x_8	0.77399112
x_9	0.700468873
x_{10}	0.754671679
x_{11}	0.715235328
x_{12}	0.744219213
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x_{19}	0.7306727
x_{20}	0.733058702
x_{21}	0.731312469



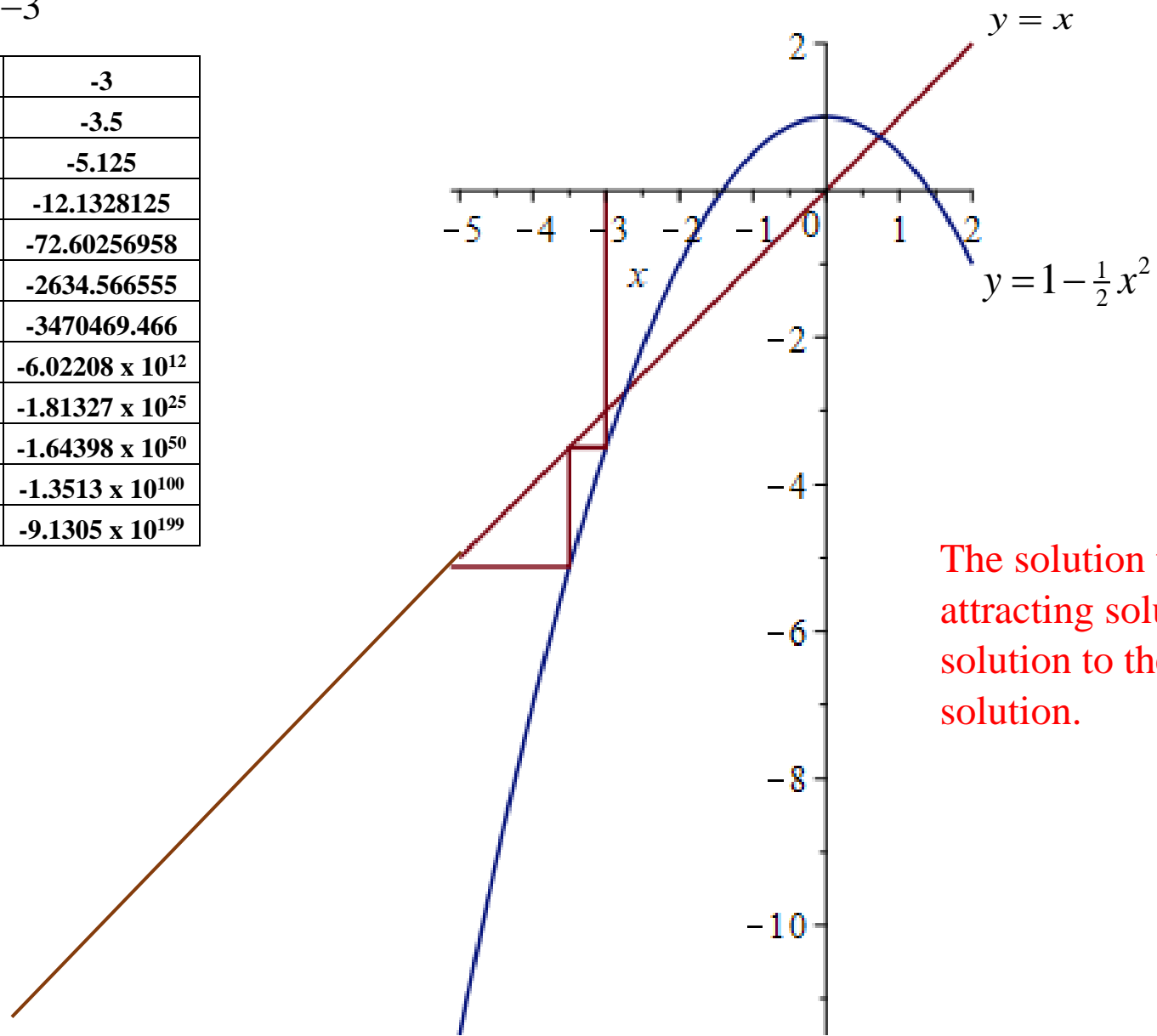
$$x_1 = -2$$

x_1	-2
x_2	-1
x_3	0.5
x_4	0.875
x_5	0.6171875
x_6	0.809539795
x_7	0.67232266
x_8	0.77399112
x_9	0.700468873
x_{10}	0.754671679
x_{11}	0.715235328
x_{12}	0.744219213
x_{13}	0.723068882
x_{14}	0.738585696
x_{15}	0.727245585
x_{16}	0.73555693
x_{17}	0.729478002
x_{18}	0.733930923
x_{19}	0.7306727
x_{20}	0.733058702
x_{21}	0.731312469



$$x_1 = -3$$

x_1	-3
x_2	-3.5
x_3	-5.125
x_4	-12.1328125
x_5	-72.60256958
x_6	-2634.566555
x_7	-3470469.466
x_8	-6.02208×10^{12}
x_9	-1.81327×10^{25}
x_{10}	-1.64398×10^{50}
x_{11}	-1.3513×10^{100}
x_{12}	-9.1305×10^{199}

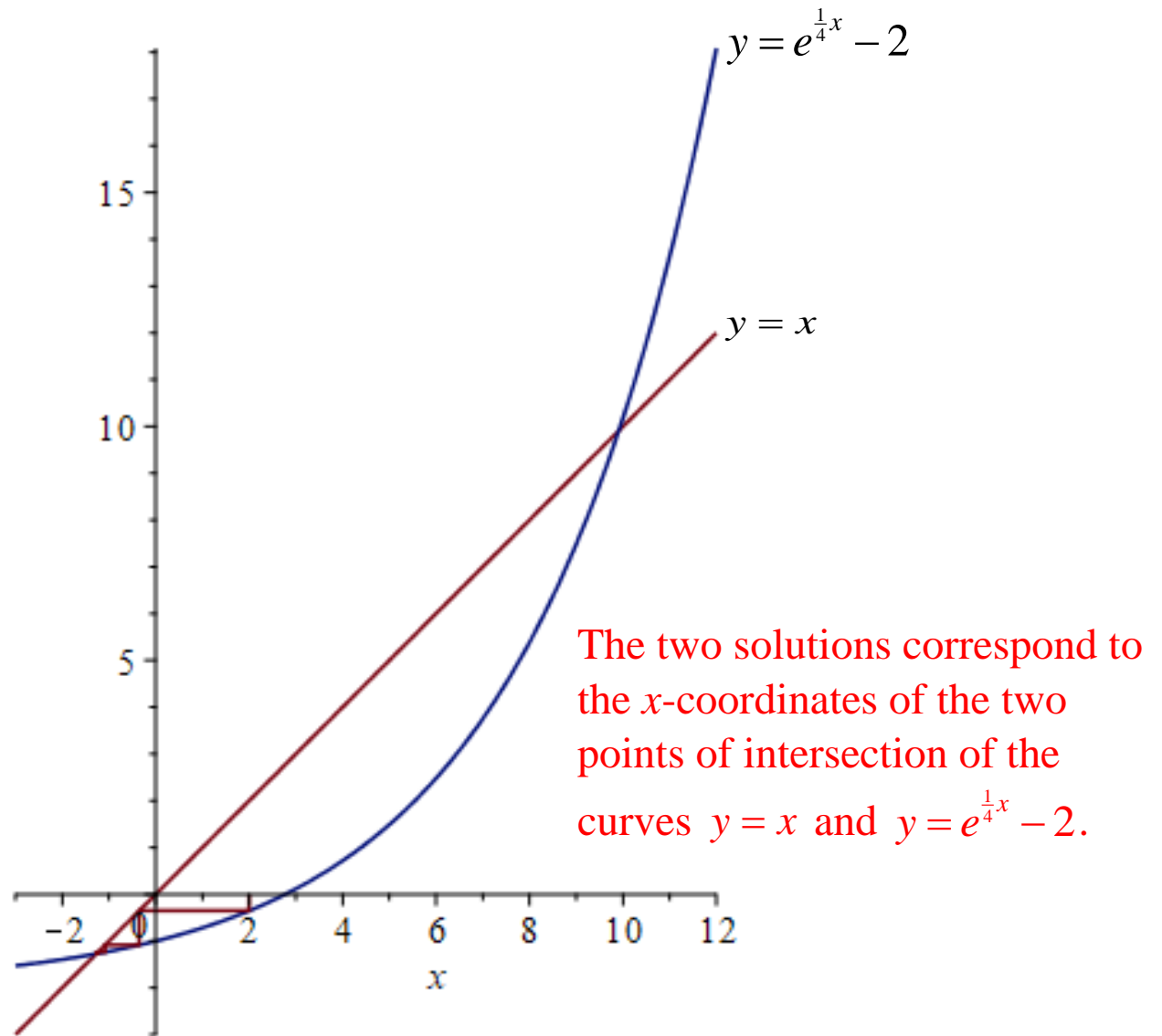


The solution to the right is an attracting solution, while the solution to the left is a repelling solution.

The equation $e^{\frac{1}{4}x} - 2 = x$ has two solutions. Classify them as attracting or repelling. Demonstrate with the starting values of $x_1 = 2, 6, 8, 10, -2$.

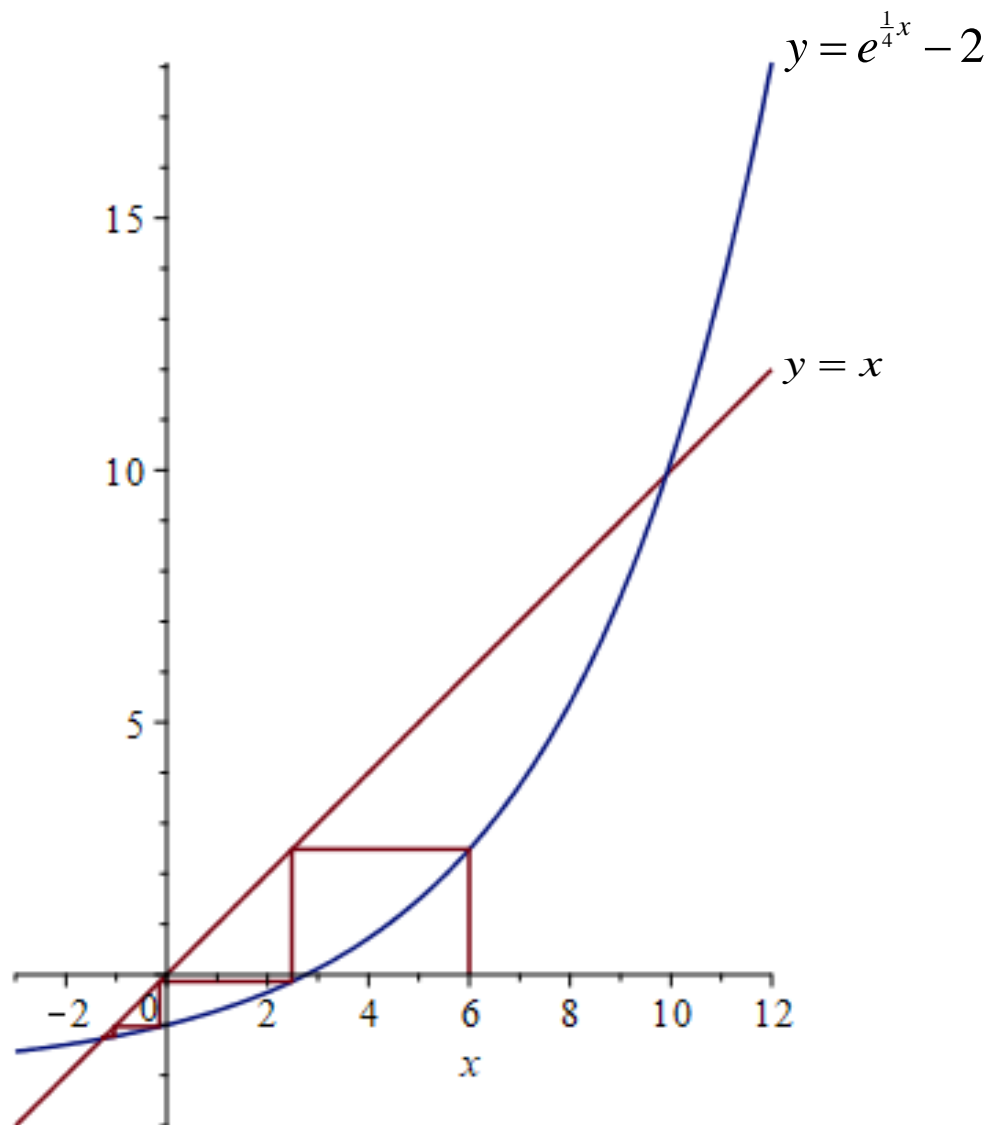
$$x_1 = 2$$

x_1	2
x_2	-0.351278729
x_3	-1.084073981
x_4	-1.237397608
x_5	-1.266075709
x_6	-1.271318781
x_7	-1.272273287
x_8	-1.272446921
x_9	-1.272478502
x_{10}	-1.272484246
x_{11}	-1.272485291
x_{12}	-1.272485481
x_{13}	-1.272485516
x_{14}	-1.272485522
x_{15}	-1.272485523
x_{16}	-1.272485523



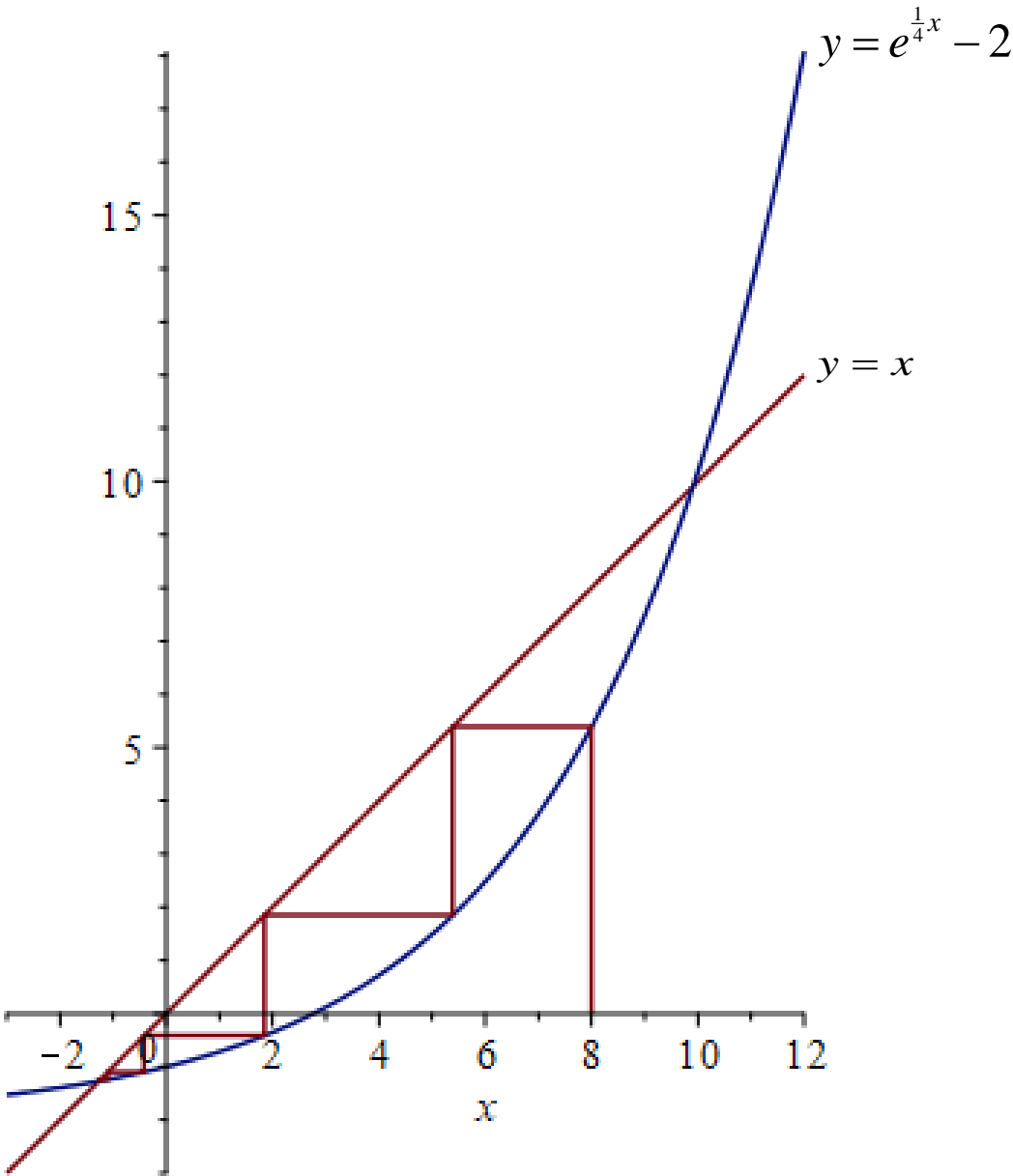
$$x_1 = 6$$

x_1	6
x_2	2.48168907
x_3	-0.140286827
x_4	-1.034463822
x_5	-1.227880506
x_6	-1.264327422
x_7	-1.271000225
x_8	-1.272215329
x_9	-1.272436379
x_{10}	-1.272476585
x_{11}	-1.272483898
x_{12}	-1.272485228
x_{13}	-1.272485469
x_{14}	-1.272485513
x_{15}	-1.272485521
x_{16}	-1.272485523
x_{17}	-1.272485523



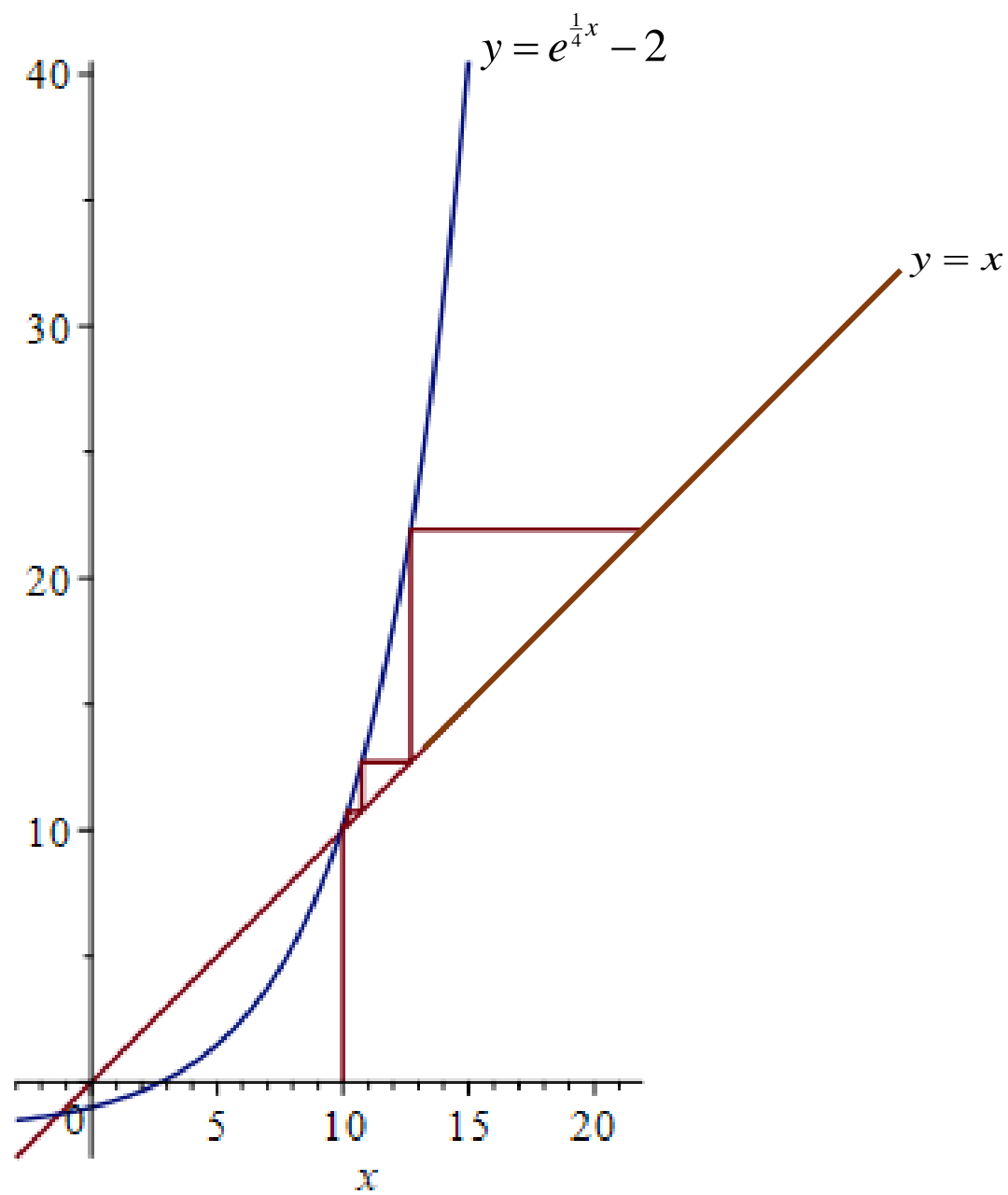
$$x_1 = 8$$

x_1	8
x_2	5.389056099
x_3	1.846886134
x_4	-0.41319663
x_5	-1.098142864
x_6	-1.240075138
x_7	-1.266566821
x_8	-1.271408241
x_9	-1.272289562
x_{10}	-1.272449881
x_{11}	-1.272479041
x_{12}	-1.272484344
x_{13}	-1.272485309
x_{14}	-1.272485484
x_{15}	-1.272485516
x_{16}	-1.272485522
x_{17}	-1.272485523
x_{18}	-1.272485523



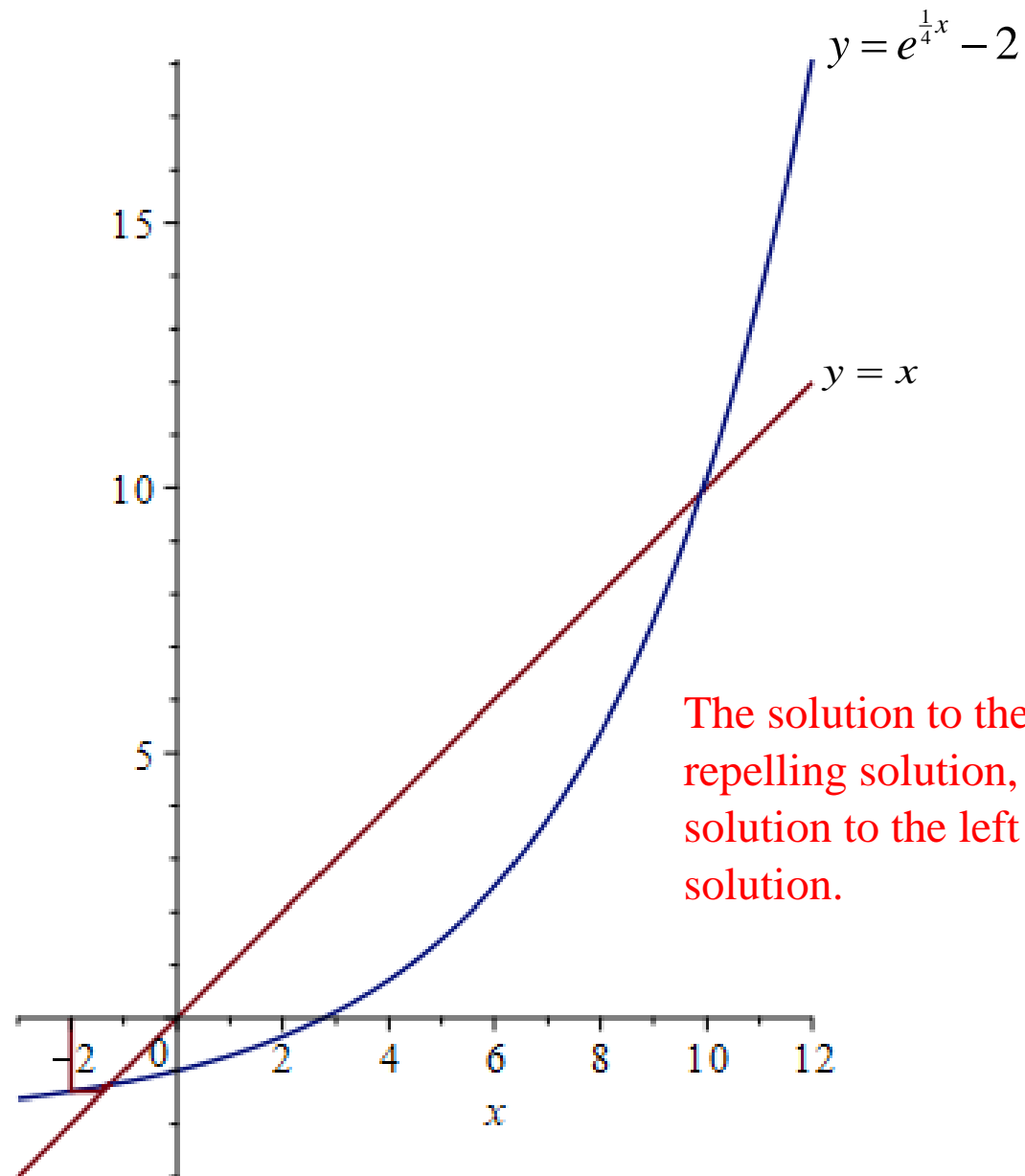
$$x_1 = 10$$

x_1	10
x_2	10.18249396
x_3	10.75117584
x_4	12.69921308
x_5	21.9221132
x_6	237.9734517
x_7	6.8808×10^{25}



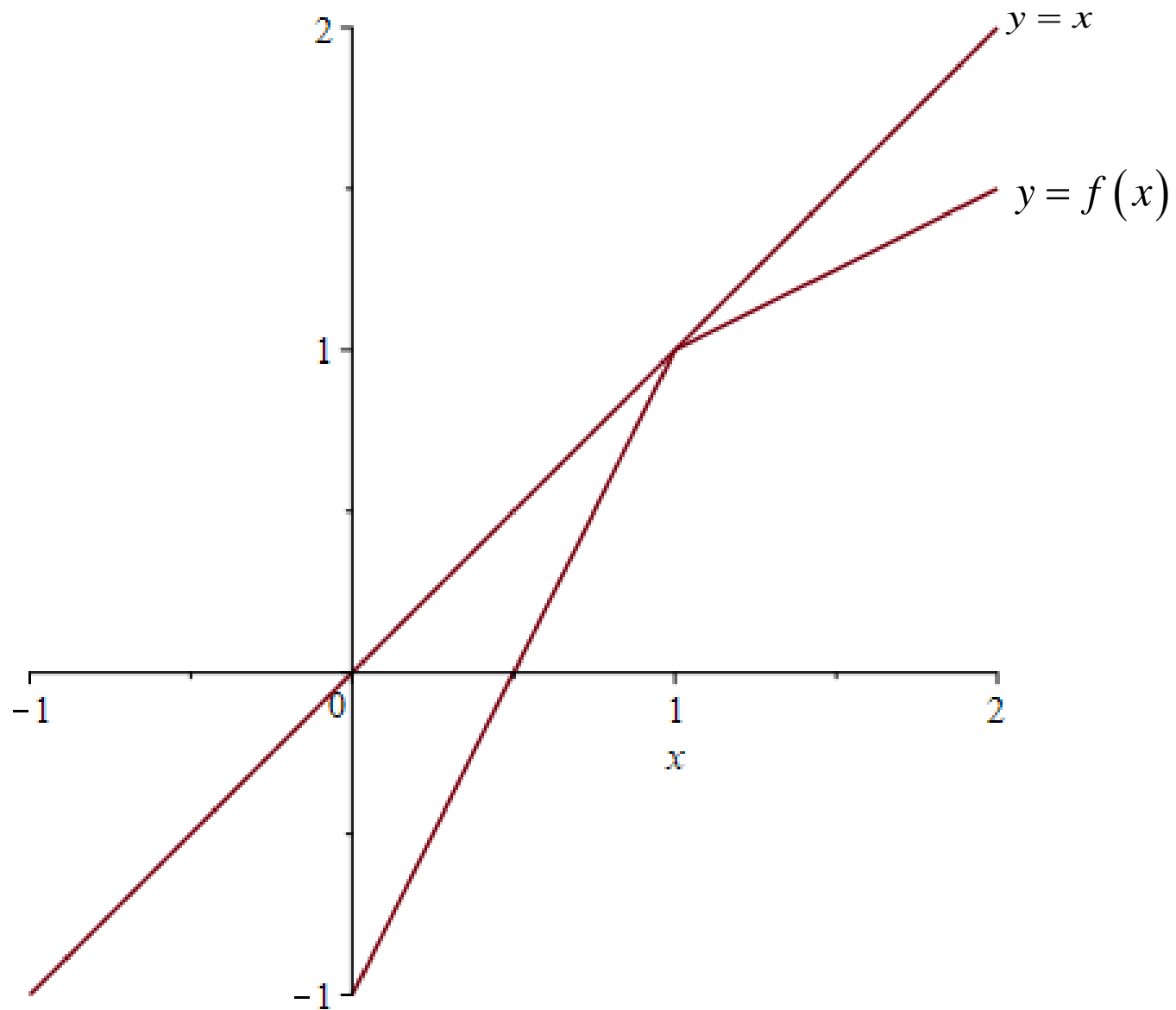
$$x_1 = -2$$

x_1	-2
x_2	-1.39346934
x_3	-1.29416045
x_4	-1.27641707
x_5	-1.27320024
x_6	-1.2726155
x_7	-1.27250916
x_8	-1.27248982
x_9	-1.27248631
x_{10}	-1.27248567
x_{11}	-1.27248555
x_{12}	-1.27248553
x_{13}	-1.27248552
x_{14}	-1.27248552



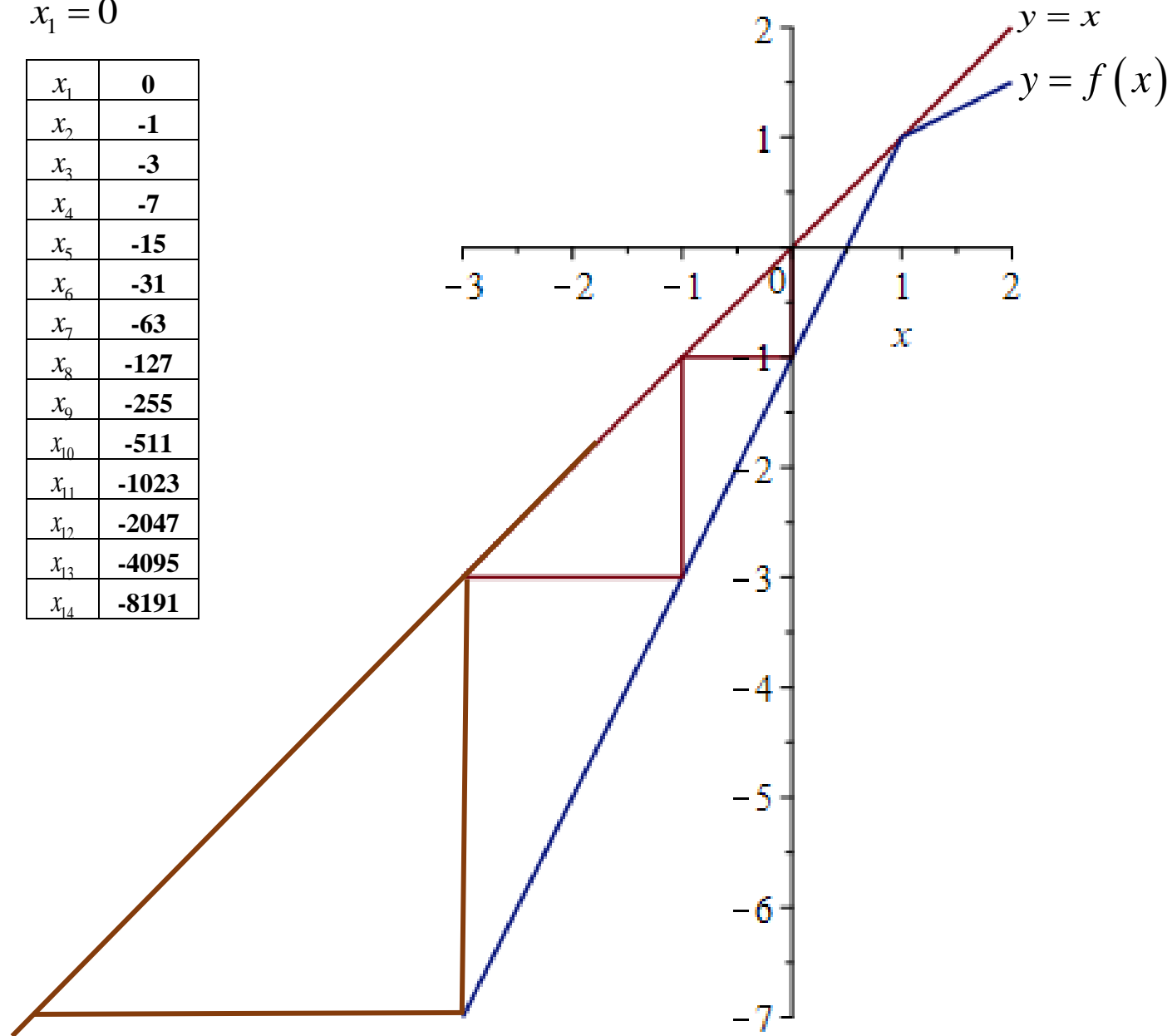
The solution to the right is a repelling solution, while the solution to the left is an attracting solution.

Other things can happen. $x = f(x) = \begin{cases} 2x-1; x \leq 1 \\ \frac{1}{2}x + \frac{1}{2}; x \geq 1 \end{cases}$ What's the exact solution? **1**



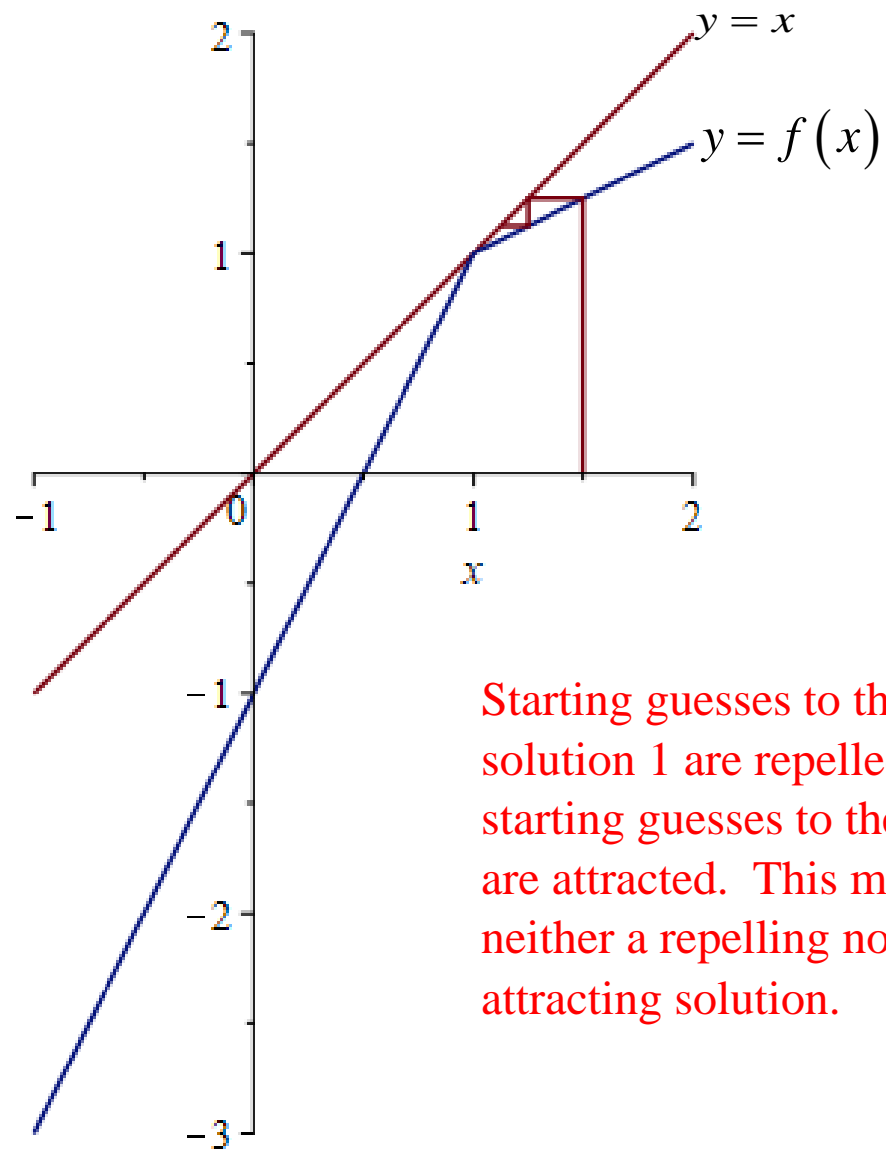
$$x_1 = 0$$

x_1	0
x_2	-1
x_3	-3
x_4	-7
x_5	-15
x_6	-31
x_7	-63
x_8	-127
x_9	-255
x_{10}	-511
x_{11}	-1023
x_{12}	-2047
x_{13}	-4095
x_{14}	-8191



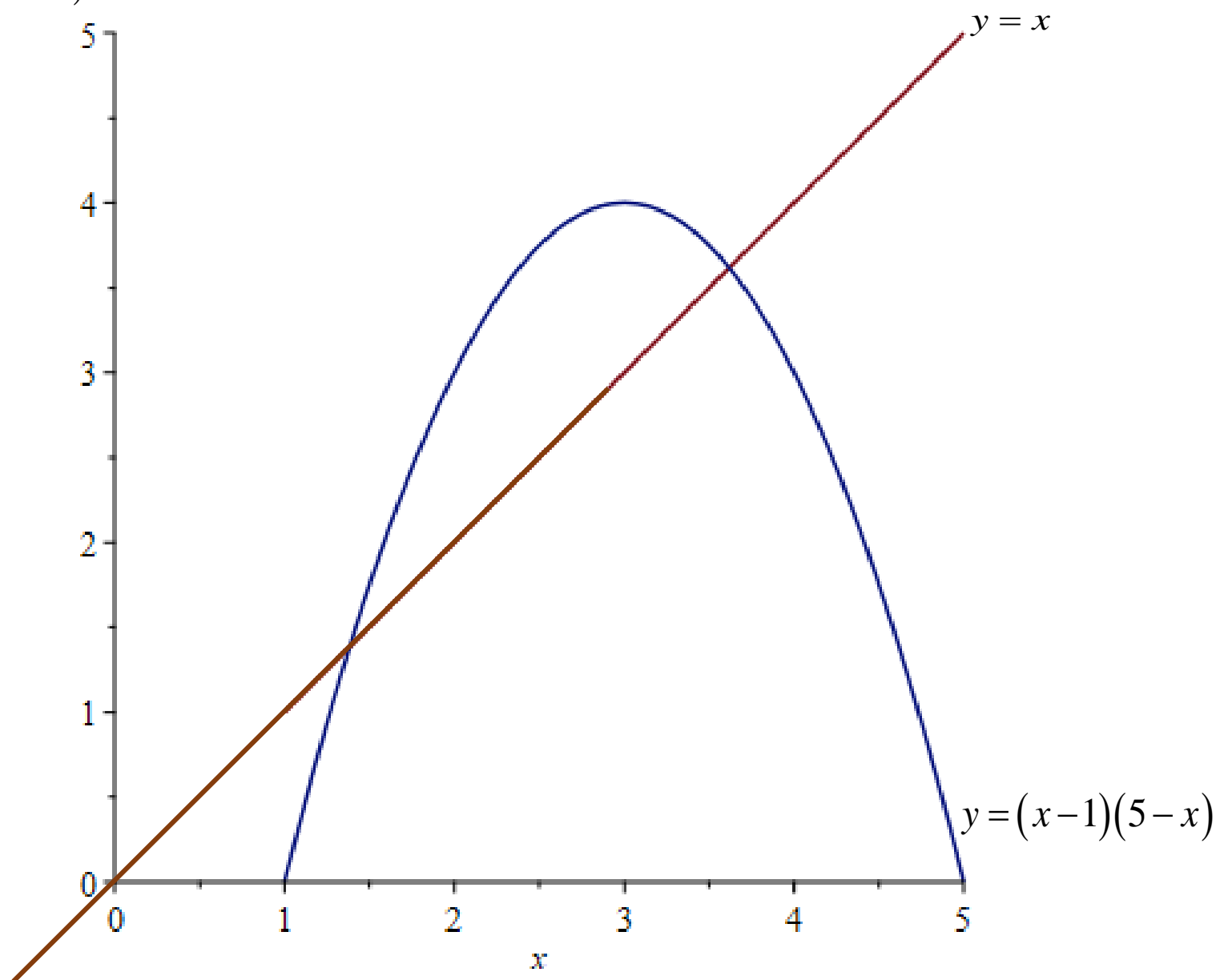
$$x_1 = 1.5$$

x_1	1.5
x_2	1.25
x_3	1.125
x_4	1.0625
x_5	1.03125
x_6	1.015625
x_7	1.0078125
x_8	1.00390625
x_9	1.001953125



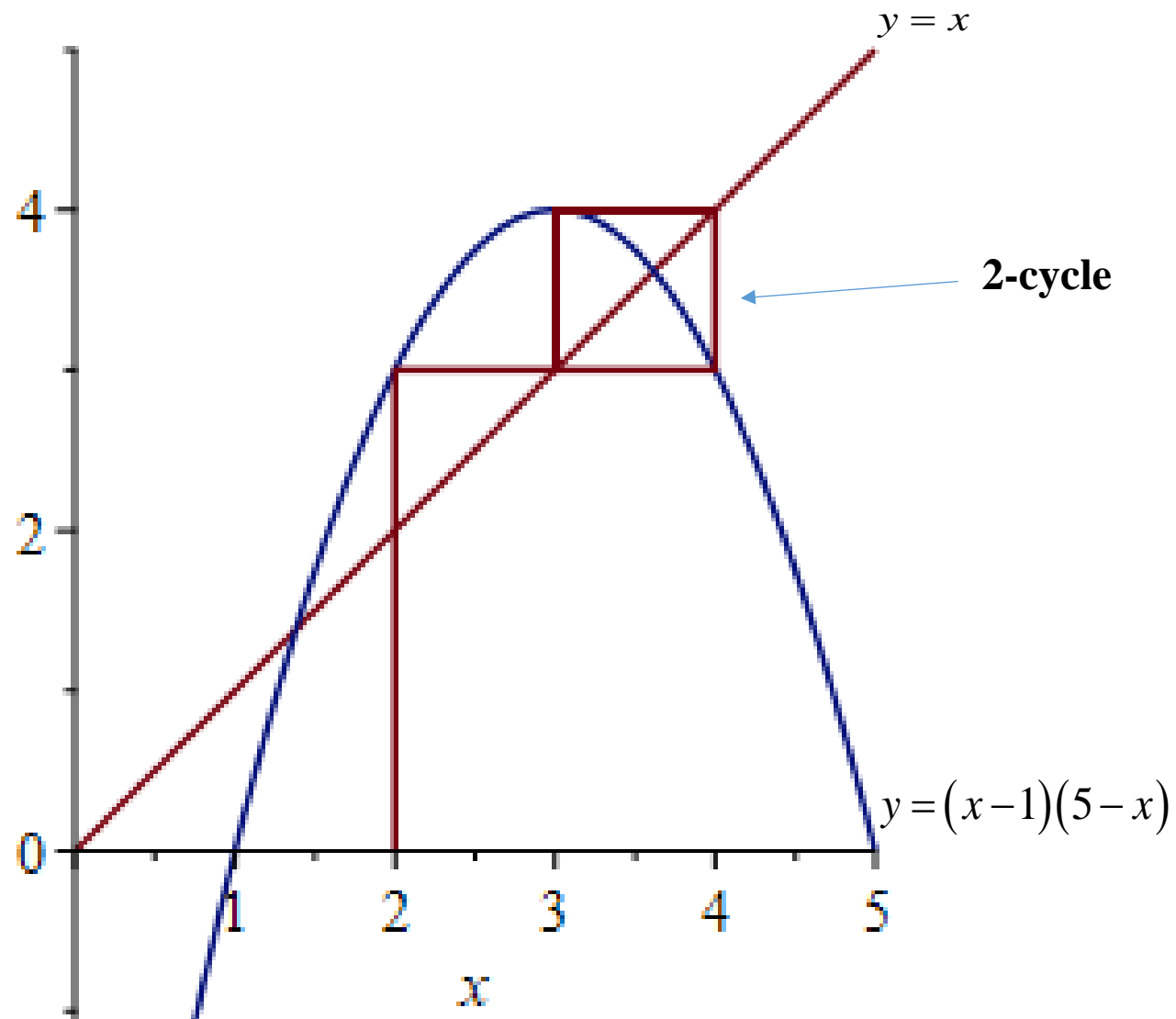
Starting guesses to the left of the solution 1 are repelled, while starting guesses to the right of 1 are attracted. This means that 1 is neither a repelling nor an attracting solution.

$$x = (x-1)(5-x)$$



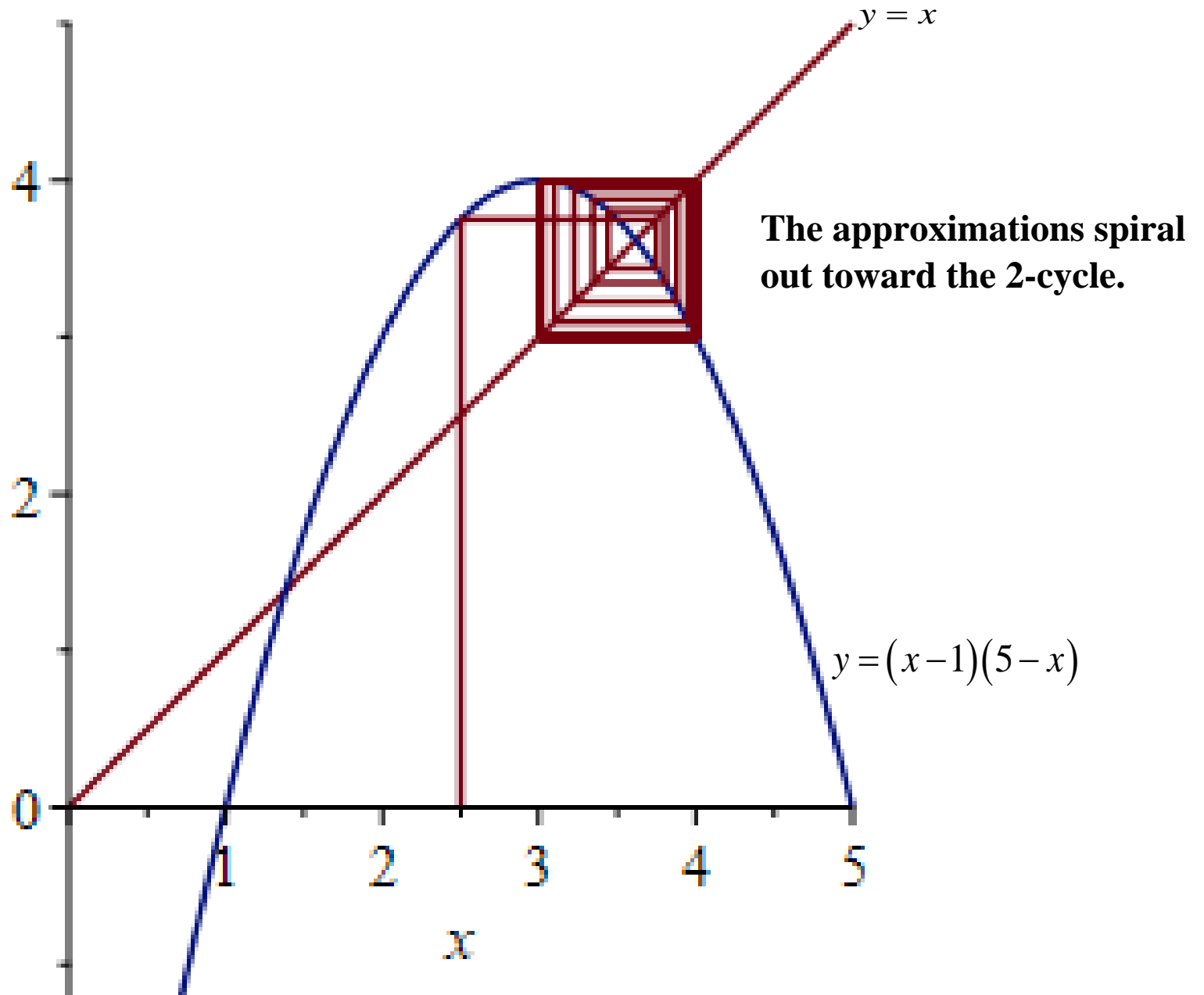
$$x_1 = 2$$

x_1	2
x_2	3
x_3	4
x_4	3
x_5	4
x_6	3
x_7	4
x_8	3
x_9	4



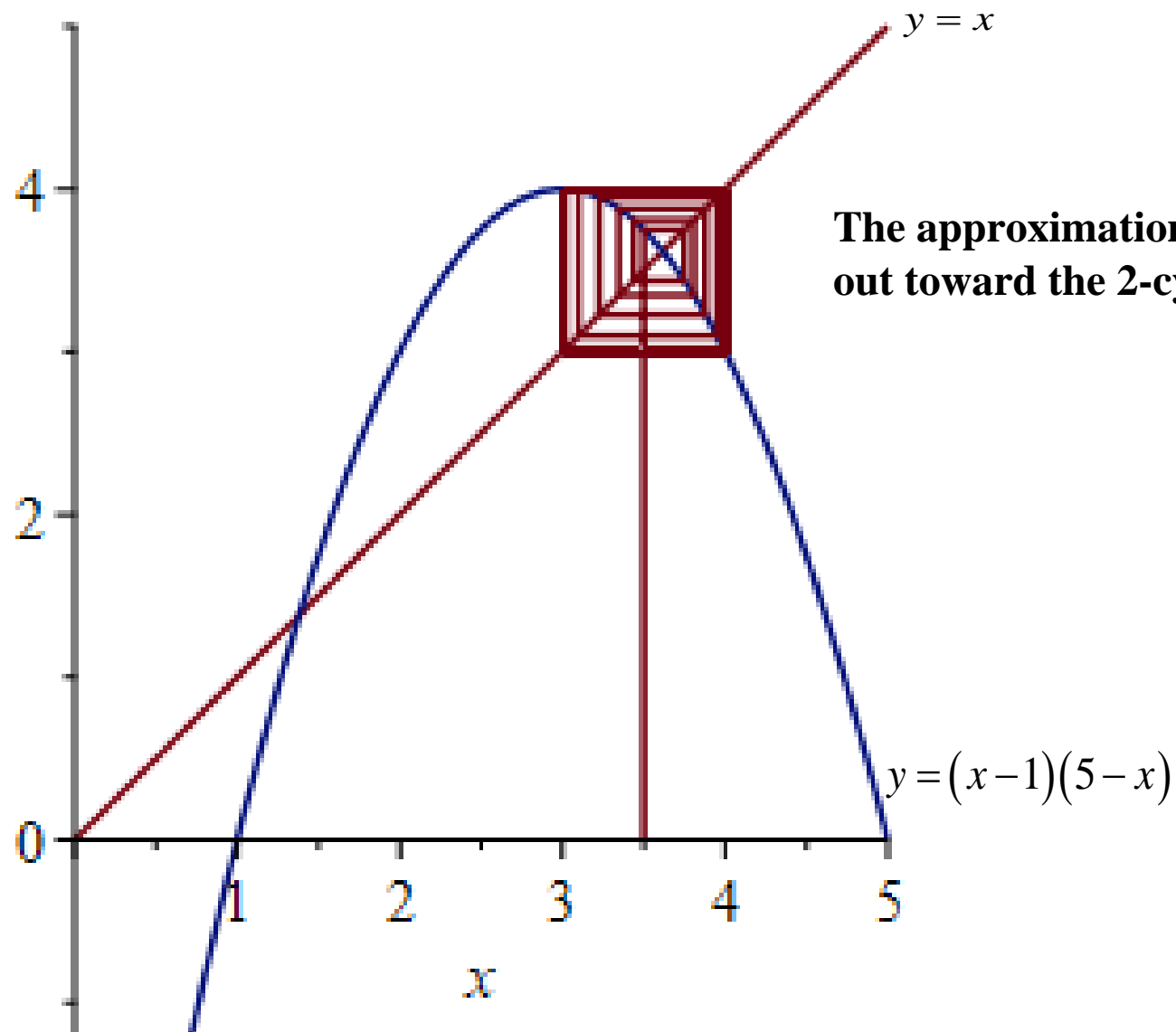
$$x_1 = 2.5$$

x_1	2.5
x_2	3.75
x_3	3.4375
x_4	3.80859375
x_5	3.34617614746093
x_6	3.88016207492910
x_7	3.22531472185650
x_8	3.94923327611473
x_9	3.09895618751650
x_{10}	3.99020767295220
x_{11}	3.01948876442659
x_{12}	3.99962018806113
x_{13}	3.00075947962064
x_{14}	3.99999942319071
x_{15}	3.00000115361826
x_{16}	3.9999999999867
x_{17}	3.00000000000266
x_{18}	4.00000000000000
x_{19}	3.00000000000000
x_{20}	4.00000000000000
x_{21}	3.00000000000000



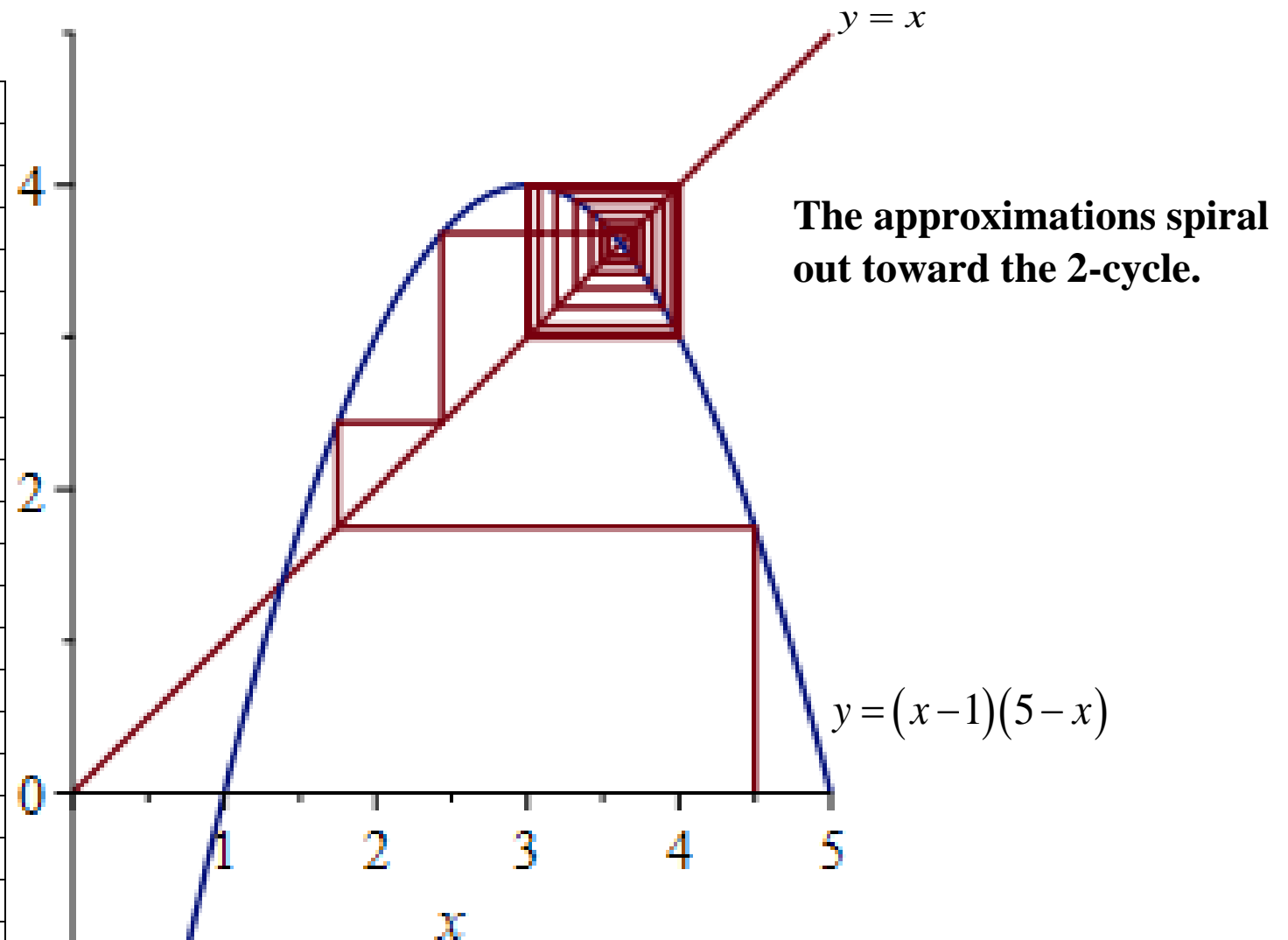
$$x_1 = 3.5$$

x_1	3.5
x_2	3.75
x_3	3.4375
x_4	3.80859375
x_5	3.34617614746093
x_6	3.88016207492910
x_7	3.22531472185650
x_8	3.94923327611473
x_9	3.09895618751650
x_{10}	3.99020767295220
x_{11}	3.01948876442659
x_{12}	3.99962018806113
x_{13}	3.00075947962064
x_{14}	3.99999942319071
x_{15}	3.00000115361826
x_{16}	3.99999999999867
x_{17}	3.000000000000266
x_{18}	4.000000000000000
x_{19}	3.000000000000000
x_{20}	4.000000000000000
x_{21}	3.000000000000000



$$x_1 = 4.5$$

x_1	4.5
x_2	1.75
x_3	2.4375
x_4	3.68359375
x_5	3.53269958496093
x_6	3.71623115218244
x_7	3.48701293664341
x_8	3.76281839954196
x_9	3.41810808932024
x_{10}	3.82518562564498
x_{11}	3.31906868322890
x_{12}	3.89819517538258
x_{13}	3.19324542691946
x_{14}	3.96265620497471
x_{15}	3.07329303102368
x_{16}	3.99462813160336
x_{17}	3.01071487982321
x_{18}	3.99988519135037
x_{19}	3.00022960411823
x_{20}	3.99999994728195
x_{21}	3.00000010543610
x_{22}	3.99999999999999
x_{23}	3.000000000000002
x_{24}	4.000000000000000



Starting guesses near the 2-cycle generate approximations that get closer to the 2-cycle, so it's called an attracting 2-cycle.

