Math 2412 Review 3(answers)

1. The decay of radium is modeled by $A(t) = A_0 e^{kt}$. If the half-life of radium is 1690 years, and you have 10 grams now, how much will be present in 50 years (rounded to three decimal places)?

$$A(t) = 10e^{kt}$$

$$5 = 10e^{k1690} \Rightarrow k = \frac{\ln\left(\frac{1}{2}\right)}{1690}$$

$$A(50) = 10e^{50 \cdot \frac{\ln\left(\frac{1}{2}\right)}{1690}} = 9.7970153\ldots = \boxed{9.797 \text{ grams}}$$

2. The logistic growth model $P(t) = \dfrac{1000}{1 + 32.33e^{-.439t}}$ represents the population (in grams) of bacteria after t hours.

a) What is the carrying capacity of the environment?

$$\boxed{1{,}000 \text{ grams}}$$

b) What is the initial population size (rounded to two decimal places)?

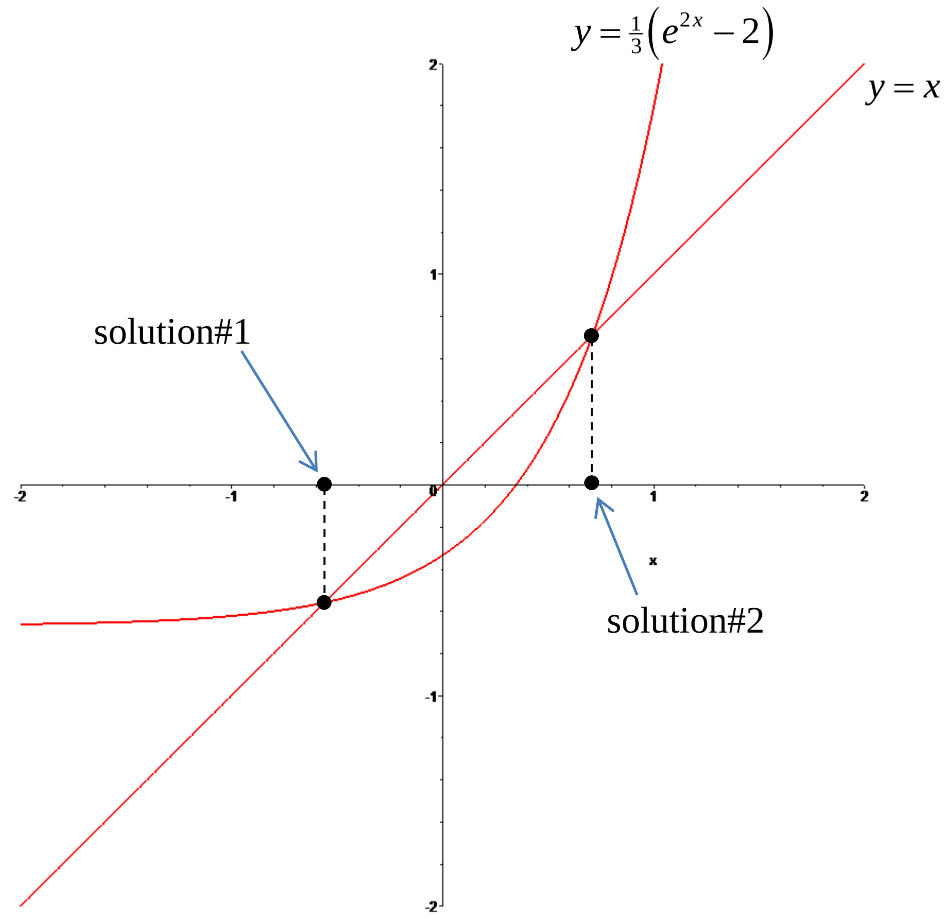
$$\frac{1000}{1 + 32.33e^{-.439(0)}} = \frac{1000}{33.33} = \boxed{30.00 \text{ grams}}$$

c) When will the population be 700 grams (rounded to two decimal places)?

$$\frac{1000}{1 + 32.33e^{-.439t}} = 700 \Rightarrow 300 = 700 \cdot 32.33e^{-.439t} \Rightarrow e^{-.439t} = \frac{300}{700 \cdot 32.33}$$

$$\Rightarrow -.439t = \ln\left(\frac{300}{700 \cdot 32.33}\right) \Rightarrow t = \frac{\ln\left(\frac{300}{700 \cdot 32.33}\right)}{-.439} = 9.848048866\ldots = \boxed{9.85 \text{ hours}}$$

3. The equation $x = \frac{1}{3}\left(e^{2x} - 2\right)$ has two solutions. In approximating them using the Method of Successive Approximation(see the graph), which solution is a repelling solution: solution#1, solution#2, or both?



Solution #2 is the repelling solution.

4. Use the method of finite differences to find a formula for generating the terms of the sequence $1, 3, 7, 13, 21, 31, 43, 57, \ldots$.

	1		3		7		13		21		31		43		57
1st difference		2		4		6		8		10		12		14	
2nd difference			2		2		2		2		2		2		

$$a_n = An^2 + Bn + C$$

$$A = \frac{2}{2} \Rightarrow \underline{A = 1}$$

$$3A + B = 2 \Rightarrow \underline{B = -1}$$

$$A + B + C = 1 \Rightarrow \underline{C = 1}$$

$$\boxed{a_n = n^2 - n + 1}$$

5. Write out the first five terms of the following sequences:

a) $a_n = \frac{n}{n+2}$

$\boxed{\left\{\frac{1}{3},\frac{1}{2},\frac{3}{5},\frac{2}{3},\frac{5}{7},\ldots\right\}}$

b) $a_1 = 1, a_{n+1} = \frac{a_n}{n}; n \geq 1$

$\boxed{\left\{1,1,\frac{1}{2},\frac{1}{6},\frac{1}{24},\ldots\right\}}$

6. Use the following formulas:

$$\sum_{k=1}^{n} k = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = 1+4+9+16+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = 1+8+27+64+\cdots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

to find the exact values of the following series:

a) $\sum_{k=1}^{200} k = \frac{200\cdot 201}{2} = \boxed{20{,}100}$

b) $\sum_{k=1}^{40} (k^2+4) = \sum_{k=1}^{40} k^2 + \sum_{k=1}^{40} 4 = \frac{40\cdot 41\cdot 81}{6} + 4\cdot 40 = \boxed{22{,}300}$

c) $\sum_{k=4}^{34} k^3 = \sum_{k=1}^{34} k^3 - \sum_{k=1}^{3} k^3 = \left(\frac{34\cdot 35}{2}\right)^2 - \left(\frac{3\cdot 4}{2}\right)^2 = \boxed{353{,}989}$

7. Consider the arithmetic sequence $5, 2, -1, -4, -7, -10, \ldots$.

a) Write a formula that will generate the terms of the sequence.

$$a_n = 5 - 3(n-1) = \boxed{8-3n}$$

b) Find the sum of the first 100 terms of this sequence.

$$S_{100} = 100\cdot\frac{5+-292}{2} = \boxed{-14{,}350}$$

8. Find the first term and the common difference for each of the following arithmetic sequences:

a) 15th term is 0 and the 40th term is -50

$$a_1 + 14d = 0$$
$$a_1 + 39d = -50$$
$$25d = -50 \Rightarrow \boxed{d = -2, a_1 = 28}$$

b) 12th term is 4 and the 18th term is 28

$$a_1 + 11d = 4$$
$$a_1 + 17d = 28$$
$$6d = 24 \Rightarrow \boxed{d = 4, a_1 = -40}$$

9. Find x so that $2x, 3x+2, 5x+3$ are consecutive terms of an arithmetic sequence.

$$3x + 2 - 2x = 5x + 3 - (3x+2) \Rightarrow x + 2 = 2x + 1 \Rightarrow x = \boxed{1}$$

10. How many terms must be added in an arithmetic sequence whose first term is 78 and whose common difference is -4 to get a sum of 702?

$$702 = n \cdot \frac{78 + 78 - 4(n-1)}{2} \Rightarrow 1404 = n(160 - 4n) \Rightarrow 4n^2 - 160n + 1404 = 0$$
$$\Rightarrow n^2 - 40n + 351 = 0 \Rightarrow (n-13)(n-27) = 0 \Rightarrow n = \boxed{13 \text{ or } 27}$$

11. Express the sum of the series $\sum_{n=1}^{1000}\left(\frac{1}{n} - \frac{1}{n+1}\right)$ as a single fraction.

$$\sum_{n=1}^{1000}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \left(\tfrac{1}{3} - \tfrac{1}{4}\right) + \cdots + \left(\tfrac{1}{1{,}000} - \tfrac{1}{1{,}001}\right) = 1 - \tfrac{1}{1{,}001} = \boxed{\tfrac{1{,}000}{1{,}001}}$$

12. Consider the geometric sequence $2, -4, 8, -16, 32, -64, \ldots$.

a) Write a formula that will generate the terms of the sequence.

$$a_n = 2(-2)^{n-1}$$

b) Find the sum of the first 12 terms of this sequence.

$$S_{12} = 2 \cdot \frac{1 - (-2)^{12}}{1 - (-2)} = 2 \cdot \frac{1 - 2^{12}}{3} = \boxed{-2730}$$

13. If $x, x+1, x+4$ are the first three terms of a geometric sequence, then what is the value of x?

$$\frac{x+1}{x} = \frac{x+4}{x+1} \Rightarrow (x+1)^2 = x^2 + 4x \Rightarrow 2x + 1 = 4x \Rightarrow 2x = 1 \Rightarrow x = \boxed{\frac{1}{2}}$$

14. Determine if the following geometric series converge or diverge. If a series converges, write what it converges to.

a) $1+-1+1+-1+1+-1+\cdots$

$$|r|=|-1|=1\geq 1 \Rightarrow \boxed{\text{Diverges}}$$

b) $1+-\frac{1}{2}+\frac{1}{4}+-\frac{1}{8}+\cdots$

$$|r|=\left|-\tfrac{1}{2}\right|=\tfrac{1}{2}<1 \Rightarrow \boxed{\text{Converges}} \text{ to } \frac{1}{1-\left(-\frac{1}{2}\right)}=\boxed{\frac{2}{3}}.$$

15. Use Mathematical Induction to prove that $3+5+7+\cdots+(2n+1)=n(n+2)$ for all natural numbers, *n*.

For $n=1$, the left-side is 3 and the right-side is $1\cdot(1+2)=3$.

Assume that the equation is true for $n=k$:

$$3+5+7+\cdots+(2k+1)=k(k+2)$$

And we'll add $\left[2(k+1)+1\right]=(2k+3)$ to both sides to get

$$\begin{aligned}3+5+7+\cdots+(2k+1)+(2k+3)&=k(k+2)+(2k+3)\\&=k^2+4k+3\\&=(k+1)(k+3)\\&=(k+1)\left[(k+1)+2\right]\end{aligned}$$

So it's true for $n=k+1$.

Therefore, $3+5+7+\cdots+(2n+1)=n(n+2)$ for all natural numbers, *n*, by Mathematical Induction.

16. Use Mathematical Induction to prove that $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{n}\right)=n+1$ for all natural numbers, *n*.

For $n=1$, the left-side is $1+\frac{1}{1}=2$ and the right-side is $1+1=2$.

Assume that the equation is true for $n=k$:

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{k}\right)=k+1$$

And we'll multiply by $\left(1+\frac{1}{k+1}\right)$ on both sides to get

$$\begin{aligned}\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{k}\right)\left(1+\frac{1}{k+1}\right)&=(k+1)\left(1+\frac{1}{k+1}\right)\\&=(k+1)+1\end{aligned}$$

So it's true for $n=k+1$.

Therefore, $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{n}\right)=n+1$ for all natural numbers, *n*, by Mathematical Induction.

17. Use Mathematical Induction to prove that $n^2 + n$ is divisible by 2 for all natural numbers *n*.

For $n = 1$, $1^2 + 1 = 2$, which is divisible by 2.

Assume it's true for $n = k$: $k^2 + k$ is divisible by 2.

$$(k+1)^2 + (k+1) = k^2 + 2k + 1 + k + 1 = k^2 + k + 2k + 2$$

$$= \underbrace{\underbrace{k^2 + k}_{\text{divisible by 2}} + \underbrace{2(k+1)}_{\text{divisible by 2}}}_{\text{divisible by 2}}$$

So it's true for $n = k + 1$.

Therefore, $n^2 + n$ is divisible by 2 for all natural numbers, *n*, by Mathematical Induction.