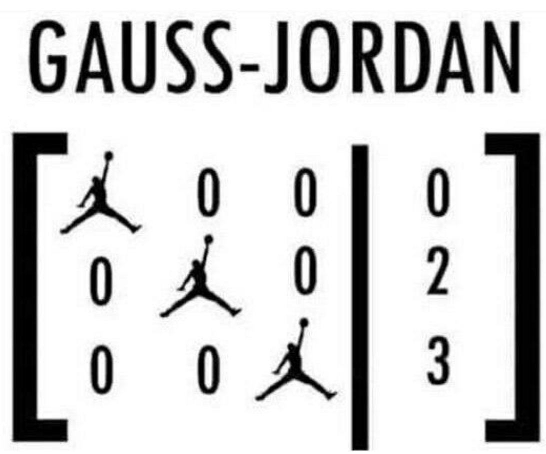
<u>Gauss-Jordan Elimination:</u>

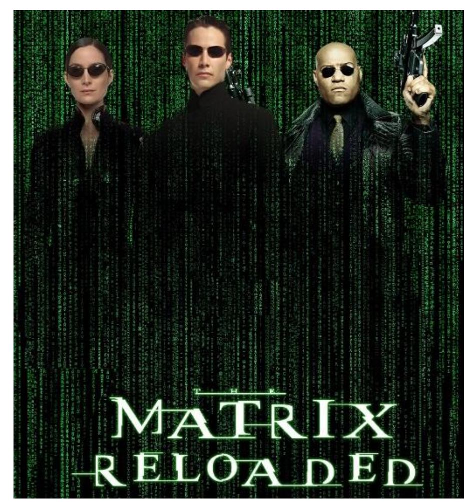
There is an extension of Gaussian Elimination called Gauss-Jordan Elimination.

In general, the goal is to use row operations to reach a matrix with the following form:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & \cdots & 0 & \alpha \\ 0 & 1 & 0 & \cdots & 0 & \beta \\ 0 & 0 & 1 & 0 & \vdots & \gamma \\ \vdots & \vdots & 0 & 1 & 0 & \delta \\ \vdots & \vdots & \vdots & \vdots & & \varepsilon \\ 0 & 0 & 0 & 0 & & \phi \end{array}\right]$$

There are as many 1's as possible on the diagonal with zeros both below the 1's and above the 1's.





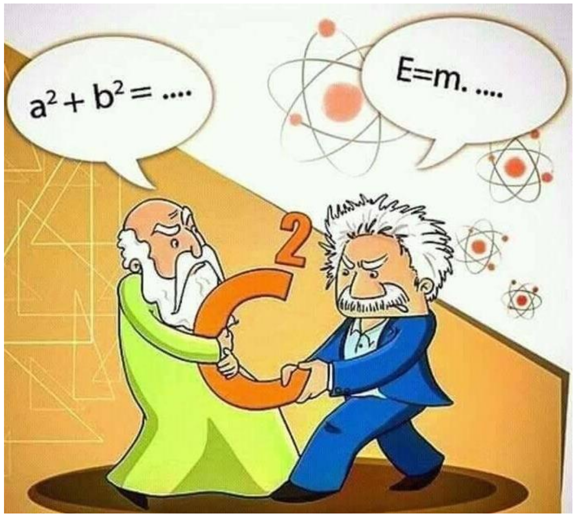
Examples:

1. $$\begin{aligned} 2x_1 + 3x_2 &= -1 \\ 3x_1 - 4x_2 &= 7 \end{aligned}$$

$$\left[\begin{array}{cc|c} 2 & 3 & -1 \\ 3 & -4 & 7 \end{array}\right] \to \left[\begin{array}{cc|c} 3 & -4 & 7 \\ 2 & 3 & -1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -7 & 8 \\ 2 & 3 & -1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -7 & 8 \\ 0 & 17 & -17 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -7 & 8 \\ 0 & 1 & -1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \end{array}\right]$$

$R_1 \leftrightarrow R_2$ $\quad -R_2 + R_1 \to R_1$ $\quad -2R_1 + R_2 \to R_2$ $\quad \frac{1}{17}R_2 \to R_2$ $\quad 7R_2 + R_1 \to R_1$

Now that the goal has been reached, you can easily see that the only solution is $x_1 = 1$ **and** $x_2 = -1$**.**



2. $$\begin{aligned} x_1 + x_2 &= 1 \\ -2x_1 - 2x_2 &= 2 \end{aligned}$$

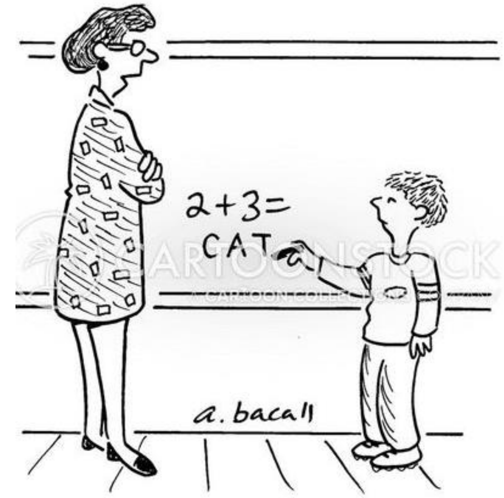
$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -2 & -2 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 4 \end{array}\right]$$

$2R_1 + R_2 \rightarrow R_2$

Now that the goal has been reached, convert the last row back into an equation:

$$0 = 4.$$

Since this is impossible, the system has no solution. If at any time in the process of reaching the goal, you get a row with zeros to the left of the bar and a non-zero number to the right, you may stop and conclude that the system has no solution.



"There's supposed to be a fine line between a guess and an educated guess."

3.

$$x_1 + x_2 = 1$$
$$3x_1 + 3x_2 = 3$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 3 & 3 & 3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$-3R_1 + R_2 \rightarrow R_2$

Now that the goal has been reached, convert the last row back into an equation:

$$0 = 0.$$

There is no contradiction, and we can't uniquely solve for the values of the variables from the first row(equation): $x_1 + x_2 = 1$**.**

When this happens, the system has infinitely many solutions, and we represent them as follows:

Let $x_2 = t$**, and substitute this into the first equation to get** $x_1 + t = 1 \Rightarrow x_1 = 1 - t$**.**

The solutions of the system are given by $x_1 = 1 - t, x_2 = t$; where t is any real number**.**

4. $$\begin{aligned} x_1 - 2x_2 + x_3 &= 2 \\ -3x_1 + x_2 + 2x_3 &= 4 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ -3 & 1 & 2 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & -5 & 5 & 10 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & 1 & -1 & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & -1 & -2 \end{array}\right]$$

$3R_1 + R_2 \to R_2$ $\qquad -\frac{1}{5}R_2 \to R_2$ $\qquad 2R_2 + R_1 \to R_1$

Now that the goal has been reached, notice that there are no contradictions, and we can't uniquely solve for the values of the variables. The system has infinitely many solutions, and we'll represent them by setting the variable furthest to the right equal to *t*: $x_3 = t$.

Substitute this into the last row(equation) to get $x_2 - x_3 = -2 \Rightarrow x_2 - t = -2 \Rightarrow x_2 = t - 2$.

Substitute the x_3 **into the first row(equation) to get**

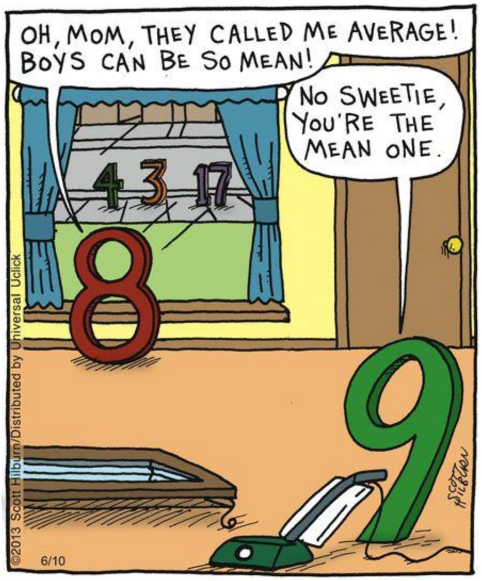
$x_1 - x_3 = -2 \Rightarrow x_1 - t = -2 \Rightarrow x_1 = t - 2$.

The solutions of the system are given by

$x_1 = t - 2, x_2 = t - 2, x_3 = t$; where t is any real number.

More Examples:

1. $$\begin{aligned} x_1 - 2x_2 &= 1 \\ 2x_1 - x_2 &= 5 \end{aligned}$$



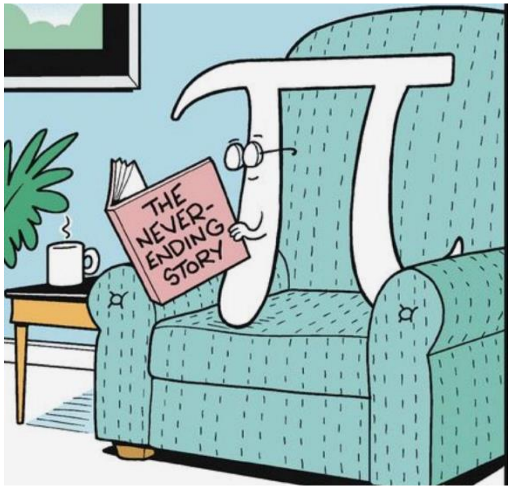
2. $$\begin{aligned} x_1 + 2x_2 &= 4 \\ 2x_1 + 4x_2 &= -8 \end{aligned}$$

"Algebra will be useful to you later in life because it teaches you to shut up and accept things that seem pointless and stupid."

3. $$\begin{aligned} 3x_1 - 6x_2 &= -9 \\ -2x_1 + 4x_2 &= 6 \end{aligned}$$

4.

$$\begin{aligned} 2x_1 + 4x_2 - 10x_3 &= -2 \\ 3x_1 + 9x_2 - 21x_3 &= 0 \\ x_1 + 5x_2 - 12x_3 &= 1 \end{aligned}$$



5.
$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 2 \\ -3x_1 - 6x_2 + 3x_3 &= -3 \end{aligned}$$

You might think that only systems with one solution or no solution are interesting or have practical applications, but systems with infinitely many solutions come up quite a bit in applications. Even though a system has infinitely many mathematical solutions doesn't mean that it has infinitely many practical solutions.

Example 1: A company wants to lease a fleet of 12 airplanes with a combined carrying capacity of 220 passengers. The three available types of planes carry 10, 15, and 20 passengers, respectively, and the leasing costs are $8,000, $14,000, and $16,000, respectively. What's the cheapest way for the company to accomplish its goal?

First, we'll figure out all the different combinations of three types of airplanes the company can lease by solving a linear system of equations:

Let $x_1 =$ **the number of 10 passenger planes,** $x_2 =$ **the number of 15 passenger planes, and** $x_3 =$ **the number of 20 passenger planes.**

$$x_1 + x_2 + x_3 = 12$$
$$10x_1 + 15x_2 + 20x_3 = 220$$

Let's solve this system using Gauss-Jordan Elimination.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 10 & 15 & 20 & 220 \end{array}\right] \underset{-10R_1+R_2\to R_2}{\to} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 5 & 10 & 100 \end{array}\right] \underset{\frac{1}{5}R_2\to R_2}{\to} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 2 & 20 \end{array}\right] \underset{-R_2+R_1\to R_1}{\to} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -8 \\ 0 & 1 & 2 & 20 \end{array}\right]$$

So the system has infinitely many solutions given by $x_1 = t - 8, x_2 = 20 - 2t, x_3 = t$**; where** ***t*** **is any real number. But,** $x_1, x_2, \text{and } x_3$ **are number of airplanes, so they have to be nonnegative whole numbers.**

So $\begin{array}{r} t-8 \geq 0 \\ 20-2t \geq 0 \\ t \geq 0 \end{array}$ **and** ***t*** **must be a whole number. This means that** $\begin{array}{l} 8 \leq t \\ t \leq 10 \\ 0 \leq t \end{array}$**, and if you combine them you get** $8 \leq t \leq 10$ **and** ***t*** **is a whole number. So instead of infinitely many solutions, we actually get three of them because** ***t*** **must be 8, 9 or 10. So the three combinations of airplanes that they can lease are** $x_1 = 0, x_2 = 4, x_3 = 8$ **and**

$x_1 = 1, x_2 = 2, x_3 = 9$ **and** $x_1 = 2, x_2 = 0, x_3 = 10$**. Now we have to determine which of these three combinations is the cheapest.**

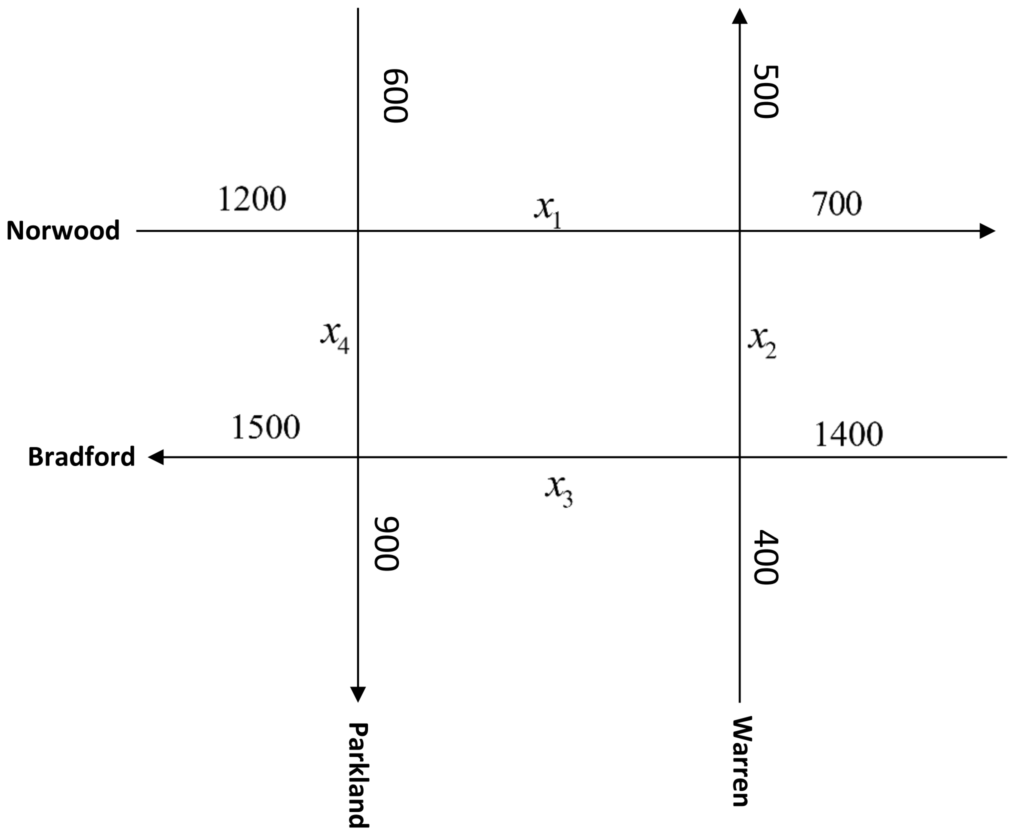
$$x_1 = 0, x_2 = 4, x_3 = 8;\ \text{cost} = \$184,000$$

$$x_1 = 1, x_2 = 2, x_3 = 9;\ \text{cost} = \$180,000$$

$$x_1 = 2, x_2 = 0, x_3 = 10;\ \text{cost} = \$176,000$$

So the cheapest way for the company to achieve its goal is to lease 2 of the 10 passenger planes and 10 of the 20 passenger planes.

Example 2: **The diagram shows the traffic flow at the intersections of four one-way streets.**



The traffic rates are in cars per hour.

In order to have smooth traffic flow, the number of cars entering an intersection must equal the number of cars leaving an intersection. This leads to four equations-one for each intersection:

Intersection	Equation
Norwood and Warren	$x_1 + x_2 = 1200$
Bradford and Warren	$x_2 + x_3 = 1800$
Bradford and Parkland	$x_3 + x_4 = 2400$
Norwood and Parkland	$x_1 + x_4 = 1800$

Here’s the augmented matrix corresponding to the system of linear equations.

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1200 \\ 0 & 1 & 1 & 0 & 1800 \\ 0 & 0 & 1 & 1 & 2400 \\ 1 & 0 & 0 & 1 & 1800 \end{array}\right]$$

The result of Gauss-Jordan Elimination is the following matrix:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1800 \\ 0 & 1 & 0 & -1 & -600 \\ 0 & 0 & 1 & 1 & 2400 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So the mathematical solution is

$x_4 = t, x_3 = 2400 - t, x_2 = t - 600, x_1 = 1800 - t,$ where t is any real #

Since the traffic flows must be nonnegative, it must be that

$$0 \le t$$
$$t \le 2400$$
$$600 \le t$$
$$t \le 1800$$

In order for all of these inequalities to be true, it must be that $600 \le t \le 1800$**. So the true solution of the system is**

$x_4 = t, x_3 = 2400 - t, x_2 = t - 600, x_1 = 1800 - t,$ where $600 \le t \le 1800$.

Here are the maximum and minimum traffic flows in the network:

Street Section	Minimum Flow	Maximum Flow
Norwood between Parkland and Warren, x_1	**0**	**1200**
Warren between Bradford and Norwood, x_2	**0**	**1200**
Bradford between Warren and Parkland, x_3	**600**	**1800**
Parkland between Bradford and Norwood, x_4	**600**	**1800**

If traffic on Warren between Bradford and Norwood is restricted to 100 cars per hour due to construction, here's the traffic flow in the rest of the system.

$$x_2 = t - 600 = 100 \Rightarrow t = 700$$

$$x_4 = 700, x_3 = 2400 - 700 = 1700, x_1 = 1800 - 700 = 1100$$

If the following tolls are charged, let's determine the least and greatest amount of money generated from the tolls per hour.

Street Section	Toll
Norwood between Parkland and Warren, x_1	**$.25**
Warren between Bradford and Norwood, x_2	**$.50**
Bradford between Warren and Parkland, x_3	**$.20**
Parkland between Bradford and Norwood, x_4	**$.15**

The total toll per hour in cents is

$$\begin{aligned}&25x_1 + 50x_2 + 20x_3 + 15x_4\\&= 25(1800 - t) + 50(t - 600) + 20(2400 - t) + 15(t)\\&= 20t + 63000;\ 600 \le t \le 1800\end{aligned}$$

So the maximum toll amount will occur for $t = 1800$**, giving a maximum toll amount of 99,000 cents or $990, and the minimum toll amount will occur for** $t = 600$**, giving a minimum toll amount of 75,000 cents or $750.**