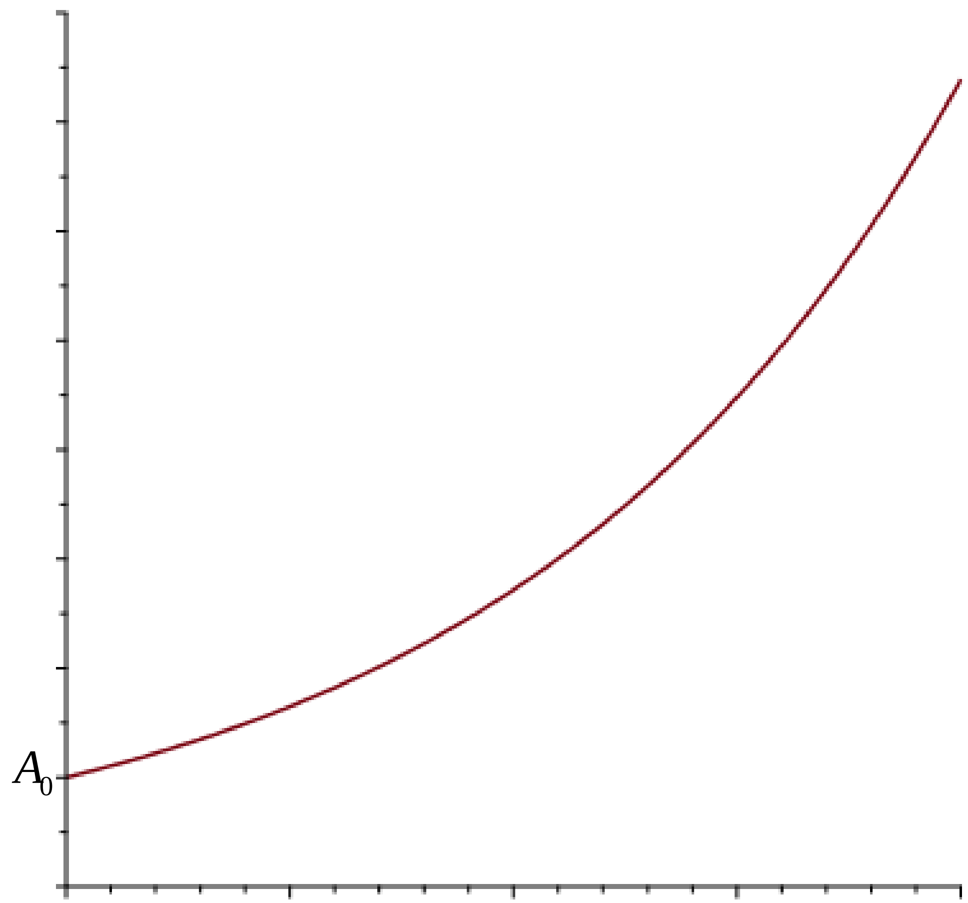
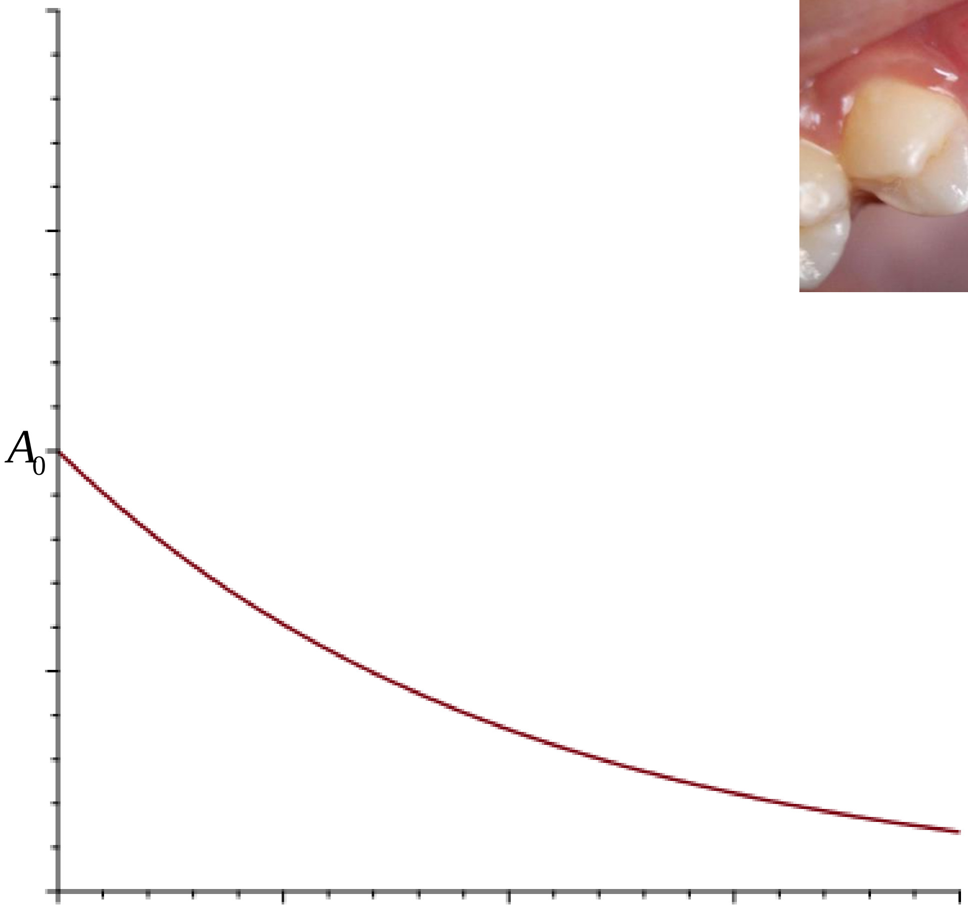
Exponential Growth and Decay Models:

$$A(t) = A_0 e^{kt}; t \geq 0, k \neq 0$$

For $k > 0$, the function models unlimited(exponential) growth, and *k* is called the growth rate.



For $k < 0$, the function models exponential decay, and k is called the decay rate.



Examples:

1. The number of bacteria in a culture is modeled by the exponential growth function $A(t) = 1000e^{.01t}$**, where** ***t*** **is measured in hours.**

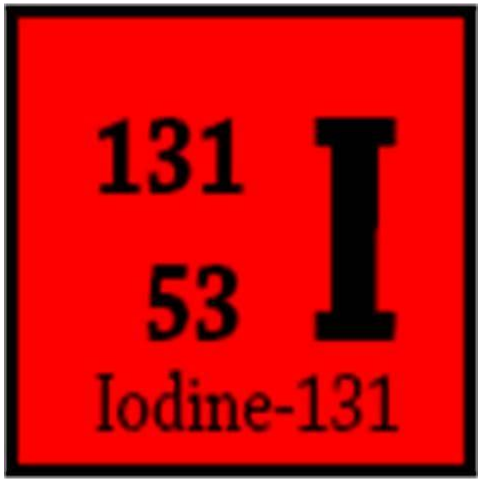
a) What is the initial number of bacteria?

b) What is the population after 4 hours?

c) When will the number of bacteria reach 1700?

d) When will the number of bacteria double?

2. The decay of Iodine-131 is modeled by the exponential decay function $A(t) = 100e^{-.087t}$, where *t* is in days, and the amount of Iodine is in grams.



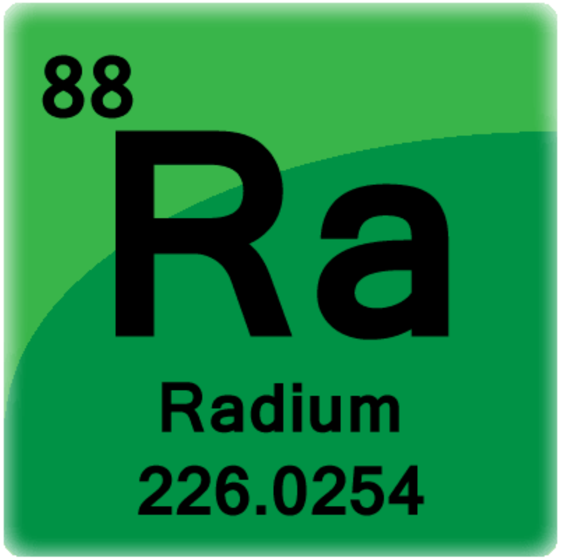
a) What is the initial amount of Iodine?

b) How much Iodine is left after 9 days?

c) When will 70 grams of Iodine be left?

d) What is the half-life of iodine-131?

3. The half-life of Radium is 1690 years. If 10 grams is present now, how much will be present in 50 years?



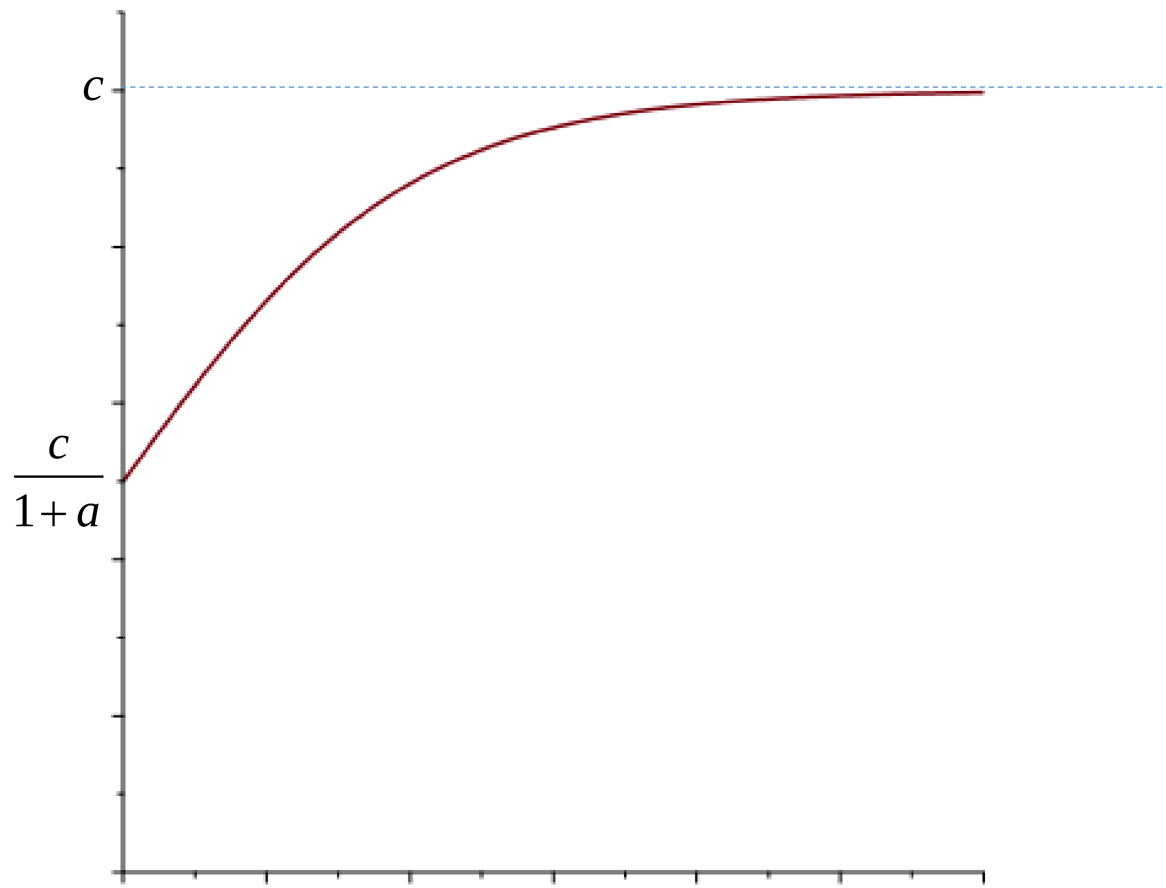
<u>Logistic Growth and Decay Models:</u>

$$P(t)=\frac{c}{1+ae^{-bt}};t\geq 0,a>0,c>0,b\neq 0$$

For $b>0$, the function models limited(logistic) growth, and *c* is called the carrying capacity.

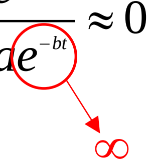
For t large, $\frac{c}{1+ae^{-bt}}\approx\frac{c}{1}=c$

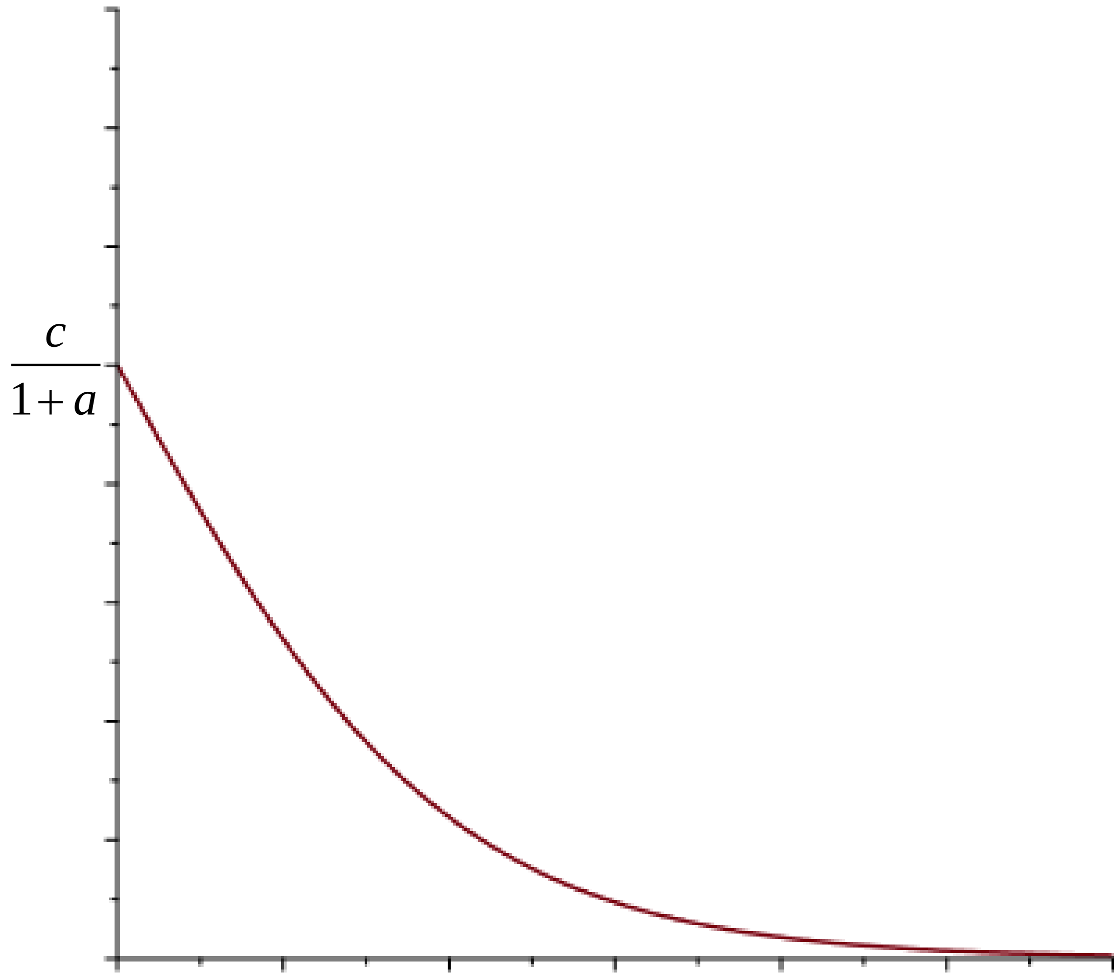
0



For $b < 0$, the function models logistic decay. Similar to exponential decay.

For t large, $\frac{c}{1 + ae^{-bt}} \approx 0$





Example:

A model for the percentage of companies using Microsoft Word is the logistic growth function $P(t) = \dfrac{99.744}{1 + 3.01e^{-.799t}}$ **where** t **is the number of years since the end of 1984.**

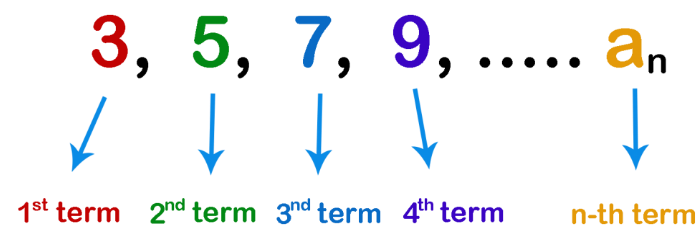
a) What was the percentage of Word users at the end of 1984?

b) What was the percentage of Word users at the end of 1990?

c) When did the percentage of Word users reach 90%?

d) What is the carrying capacity percentage for Word users?

Sequences:



A sequence is an ordered list of infinitely many numbers.

They can be represented by implying a pattern*(partial list)*, giving a direct formula, or giving a recursive formula.

Implying a pattern(partial list): ***{Determine the next two terms of each sequence.}***

$\{a_n\} = \{1,2,3,4,\dots\}$

$\{b_n\} = \{1,\tfrac{1}{2},\tfrac{1}{3},\tfrac{1}{4},\dots\}$

$\{c_n\} = \{1,-1,1,-1,\dots\}$

$\{d_n\} = \{1,-2,3,-4,5,-6,\dots\}$

$\{e_n\} = \{1,-\tfrac{1}{2},\tfrac{1}{3},-\tfrac{1}{4},\tfrac{1}{5},-\tfrac{1}{6},\dots\}$

Direct Formula: **Unless stated otherwise, assume the starting subscript value is 1.**

{Determine the first five terms of each sequence.}

$a_n = n$

$b_n = \frac{1}{n}$

$c_n = (-1)^{n+1}$

$d_n = (-1)^{n+1} n$

$$e_n = \frac{(-1)^{n+1}}{n}$$

<u>Recursive Formula:</u> ***{Determine the first five terms of each sequence.}***

$$a_1 = 1, a_{n+1} = 1 + a_n; n \geq 1$$

$$b_1 = 1, b_{n+1} = \frac{1}{1 + \frac{1}{b_n}}; n \geq 1$$

$$c_1 = 1, c_{n+1} = -c_n; n \geq 1$$

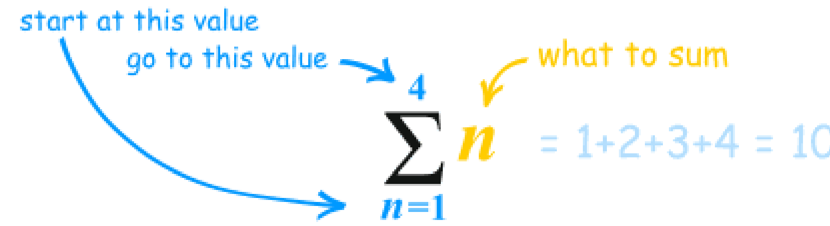
$$d_1 = 1, d_{n+1} = -d_n + (-1)^n; n \geq 1$$

$$e_1 = 1, e_{n+1} = \frac{1}{-\frac{1}{e_n} + \left(-1\right)^n}; n \geq 1$$

$$f_1 = 1, f_2 = 1, f_{n+2} = f_{n+1} + f_n; n \geq 1$$

{Fibonacci}

$$\boldsymbol{F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}}$$



<u>Sigma or Summation Notation:</u>

$a_1 + a_2 + a_3 + \cdots + a_n$ **can be abbreviated as** $\sum_{k=1}^{n} a_k$**. In other words,**

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n.$$

Examples:

1. Expand $\sum_{k=1}^{3} k^2$**.**

2. Expand $\sum_{k=1}^{4} (-1)^k$**.**

3. Compress $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$**.**

$\sum_{k=1}^{n} a_k$ **is considered to be a sum of a portion of the terms of the sequence** $\{a_1, a_2, a_3, \ldots\}$ **, and is sometimes referred to as a *<u>finite series</u>*.**

<u>Properties of Finite Series:</u> If $\{a_n\}$ and $\{b_n\}$ are sequences and c is any real number, then

1. $\sum_{k=1}^{n}(ca_k) = c\sum_{k=1}^{n} a_k$

Why? $\sum_{k=1}^{n}(ca_k) = (ca_1 + ca_2 + \cdots + ca_n) =$

2. $\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

Why? $\sum_{k=1}^{n}(a_k + b_k) = (a_1 + b_1) + \cdots + (a_n + b_n) =$

3. $$\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

Why? $$\sum_{k=1}^{n}(a_k - b_k) = (a_1 - b_1) + \cdots + (a_n - b_n) =$$

4. $$\sum_{k=j+1}^{n} a_k = \sum_{k=1}^{n} a_k - \sum_{k=1}^{j} a_k \quad \text{for } 1 \le j \le n-1$$

Why? $$\sum_{k=1}^{n} a_k - \sum_{k=1}^{j} a_k = (a_1 + a_2 + \cdots + a_j + a_{j+1} + \cdots + a_n) - (a_1 + a_2 + \cdots + a_j) =$$

<u>Special Formulas for Finite Series:</u>

$$\sum_{k=1}^{n} c = \underbrace{c + c + c + \cdots + c}_{(n \text{ terms})} = nc$$

Examples:

1. $\sum_{k=1}^{5} 2$

2. $\sum_{k=1}^{5,000} 3$

3. $\sum_{k=12}^{200} 2$ $\left\{ \sum_{k=12}^{200} 2 = \sum_{k=1}^{200} 2 - \sum_{k=1}^{11} 2 \right\}$

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n$$

(the sum of the first n counting numbers)

Let $S = 1 + 2 + 3 + \cdots + (n-1) + n$**.** **Then also,** $S = n + (n-1) + \cdots + 2 + 1$**.**

$$\begin{array}{rcccccccccccc} S = & 1 & + & 2 & + & 3 & + \cdots + & (n-1) & + & n \\ +S = & n & + & (n-1) & + & (n-2) & + \cdots + & 2 & + & 1 \\ \hline 2S = & & & & & & & & & \end{array}$$

So $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$**.**

Examples:

1. $\sum_{k=1}^{100} k$

2. $\sum_{k=1}^{100} (k+2)$

3. $\sum_{k=1}^{100} (2k)$

4. $\sum_{k=11}^{100} k$ $\left\{ \sum_{k=11}^{100} k = \sum_{k=1}^{100} k - \sum_{k=1}^{10} k \right\}$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2$$ ***(the sum of the squares of the first n counting numbers)***

$$\sum_{k=1}^{n} (k+1)^3 - \sum_{k=1}^{n} k^3 = \left[2^3 + 3^3 + \cdots + (n+1)^3\right] - \left[1^3 + 2^3 + \cdots + n^3\right] = (n+1)^3 - 1$$

And

$$\sum_{k=1}^{n} (k+1)^3 - \sum_{k=1}^{n} k^3 = \sum_{k=1}^{n} \left[(k+1)^3 - k^3\right] = \sum_{k=1}^{n} \left(k^3 + 3k^2 + 3k + 1 - k^3\right) = \sum_{k=1}^{n} \left(3k^2 + 3k + 1\right)$$

$$= 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = 3\sum_{k=1}^{n} k^2 + \frac{3n(n+1)}{2} + n$$

So

$$3\sum_{k=1}^{n} k^2 + \frac{3n(n+1)}{2} + n = (n+1)^3 - 1.$$

This means that $$\sum_{k=1}^{n} k^2 = \frac{(n+1)^3 - 1 - \frac{3n(n+1)}{2} - n}{3} = \frac{n(n+1)(2n+1)}{6}.$$

Examples:

1. $\sum_{k=1}^{12} k^2$

2. $\sum_{k=1}^{12} (k^2 - 4)$

3. $\sum_{k=1}^{12} (2k^2 - k + 1)$