<u>The Method of Finite Differences:</u>

Sometimes it's possible to find a nice formula for the terms of a sequence. Given a sequence, you can form a new sequence by subtracting each term from the term that follows it. The new sequence is called the sequence of first differences.

Example: For the sequence $\{a_n\} = \{2,5,8,11,14,17,20,\ldots\}$

a_n	**2**		**5**		**8**		**11**		**14**		**17**		**20**
1st differences		**3**											

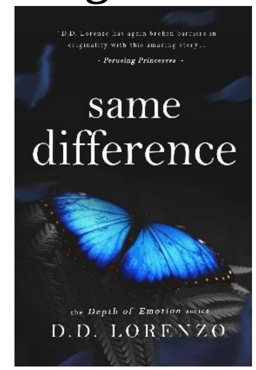
If the sequence of first differences is a constant sequence, then the terms of the original sequence can be generated by a linear formula: $a_n = An + B$**. Find the formula for this example, assuming that the sequence of first differences remains constant.**

$$a_n = An + B$$

$(n=1)$ $\quad A + B = 2$

$(n=2)$ $\quad 2A + B = 5$

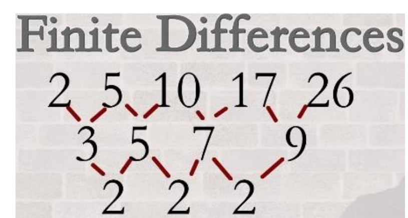
The differences of the sequence of first differences can also be calculated to get the sequence of second differences.



Example: For the sequence $\{a_n\} = \{0,4,10,18,28,40,54,70,88,\ldots\}$

a_n	0		4		10		18		28		40		54		70		88
1st differences		4		6		8		10		12							
2nd differences			2		2		2										

If the sequence of second differences is a constant sequence, then the terms of the original sequence can be generated by a quadratic formula: $a_n = An^2 + Bn + C$**. Find the formula for this example, assuming that the sequence of second differences remains constant.**



$$a_n = An^2 + Bn + C$$

$$(n=1) \quad A + B + C = 0$$

$$(n=2) \quad 4A + 2B + C = 4$$

$$(n=3) \quad 9A + 3B + C = 10$$

Similar results hold for sequences with constant third differences, and fourth differences, and …. See the link Finite Differences.

Let's find a formula for the sequence $\{a_n\} = \{0, 7, 26, 63, 124, 215, 342, 511, \ldots\}$**. See the link Finite Difference XL.**

n	1	2	3	4	5	6	7	8
sequence values	0	7	26	63	124	215	342	511
first differences		7	19	37	61	91	127	169
second differences			12	18	24	30	36	42
third differences				6	6	6	6	6

So the terms of the sequence can be generated by a cubic formula $a_n = An^3 + Bn^2 + Cn + D$**.**

$$a_n = An^3 + Bn^2 + Cn + D$$

$$(n=1) \quad A + B + C + D = 0$$

$$(n=2) \quad 8A + 4B + 2C + D = 7$$

$$(n=3) \quad 27A + 9B + 3C + D = 26$$

$$(n=4) \quad 64A + 16B + 4C + D = 63$$

Let's use Finite Differences to find a formula for the sum of the first *n* counting numbers:

a_n	1		$1+2$		$1+2+3$		$1+2+3+4$		$1+2+3+4+5$
1st differences		2		3		4		5	
2nd differences			**1**		**1**		**1**		

$$a_n = An^2 + Bn + C$$

$$A + B + C = 1$$

$$4A + 2B + C = 3$$

$$9A + 3B + C = 6$$

Let's use Finite Differences to find a formula for the sum of the squares of the first *n* counting numbers:

a_n	1^2		1^2+2^2		$1^2+2^2+3^2$		$1^2+2^2+3^2+4^2$		$1^2+2^2+3^2+4^2+5^2$
1st differences		2^2		3^2		4^2		5^2	
2nd differences			**5**		**7**		**9**		
3rd differences				**2**		**2**			

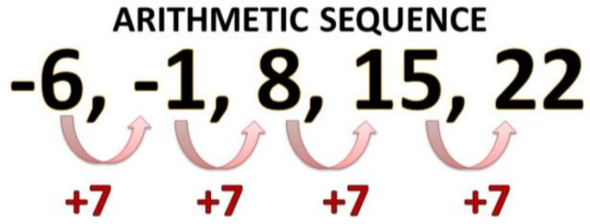
$$a_n = An^3 + Bn^2 + Cn + D$$

$$A + B + C + D = 1$$
$$8A + 4B + 2C + D = 5$$
$$27A + 9B + 3C + D = 14$$
$$64A + 16B + 4C + D = 30$$

Be careful! There are some assumptions in the Method of Finite Differences.

For example, for this sequence $\{1,2,3,\ldots\}$**, what's the fourth term?**

$$a_n = (n-1)(n-2)(n-3) + n$$



Arithmetic Sequences:

It's a sequence of the form $\{a_1, a_1 + d, a_1 + 2d, a_1 + 3d, \ldots\}$**, where consecutive terms differ by the same value,** ***d*****, called the common difference.**

A formula for the terms of an arithmetic sequence is $a_n =$ __________________ **.**

Examples:

1. For the arithmetic sequence $\{2, 8, 14, 20, 26, \ldots\}$**,**

a) Find the common difference.

b) Find a formula for the value of the n^{th} **term of the sequence.**

c) Find the 20^{th} **term in the sequence.**

d) Is 2146 a number in this sequence?

e) Is 1034 a number in this sequence?

2. Find a formula for the n^{th} term of the arithmetic sequence with 4^{th} term of 3 and 20^{th} term of 35.

$a_n = a_1 + (n-1)d$ **, so** $a_1 + 3d = 3$

$$a_1 + 19d = 35$$

3. How many numbers are in the partial arithmetic sequence $\{8,5,2,-1,-4,\ldots,-295\}$ **?**

$a_n = 8 + (n-1)(-3)$

4. Find *x* so that $2x, 3x+2, 5x+3$ **are consecutive terms in an arithmetic sequence.**

$(3x+2) - 2x = d$ **and** $(5x+3) - (3x+2) = d$

The sum of the first n terms of an arithmetic sequence:

$$\begin{aligned}
S_n &= a_1 + a_2 + a_3 + \cdots + a_n \\
&= a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + \left[a_1 + (n-1)d\right] \\
&= na_1 + \left[1 + 2 + 3 + \cdots + (n-1)\right]d \\
&= na_1 + \left[\frac{n(n-1)}{2}\right]d \\
&= n\left[a_1 + \frac{(n-1)d}{2}\right] \\
&= n\left[\frac{a_1 + \left[a_1 + (n-1)d\right]}{2}\right] \\
&= n\left[\frac{a_1 + a_n}{2}\right]
\end{aligned}$$

Yes, it's just *n* times the average of the first and last terms!

Find the following sums of partial arithmetic sequences:

1. $1+3+5+\cdots+(2n-1)$

2. $2+5+8+\cdots+41$

3. $7+1-5-11-\cdots-299$

4. $\sum_{n=1}^{90}(3-2n)$

5. The sum of the first 46 terms of the sequence $\{2,-1,-4,-7,\ldots\}$

How many terms must be added in an arithmetic sequence whose first term is 78 and whose common difference is -4 to get a sum of 702?

$$n\left[\frac{\underset{a_1}{78}+\underbrace{78+(n-1)(-4)}_{a_n}}{2}\right]=702$$

$$n\left[\frac{-4n+160}{2}\right]=702$$

$$n(-2n+80)=702$$

$$-2n^2+80n=702$$

$$2n^2-80n+702=0$$

$$n^2-40n+351=0$$

1	**351**
3	**117**
9	**39**
13	**27**

Factor or quadratic formula!