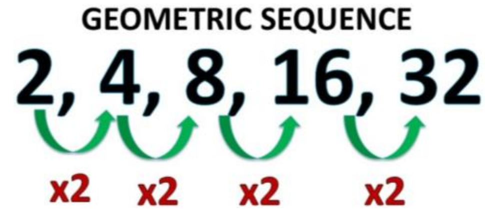

Geometric Sequences:

It's a sequence of the form $\{a_1, a_1r, a_1r^2, a_1r^3, \ldots\}$**, where each term beyond the first is the factor,** ***r*****, times the previous term.** ***r*** **is called the common ratio, and** $a_1 \neq 0$**.**

A formula for the terms of a geometric sequence is $a_n =$ **.**

Examples:

1. For the geometric sequence $\left\{1, \frac{2}{5}, \frac{4}{25}, \frac{8}{125}, \frac{16}{625}, \ldots\right\}$**,**

a) Find the common ratio.

b) Find a formula for the value of the ***n*****th term of the sequence.**

c) Find the 8th term in the sequence. **d) Is** $\frac{1{,}024}{9{,}765{,}625}$ **a number in this sequence?**

e) Is $\frac{4{,}096}{48{,}828{,}125}$ **a number in this sequence?**

2. Find a formula for the n^{th} term of the geometric sequence of real numbers with 3^{rd} term of 3 and 6^{th} term of 24.

$$?\ ,\ ?\ ,\ 3\ ,\ ?\ ,\ ?\ ,\ 24$$

3. Find x so that $x-1, x, x+2$ are consecutive terms in a geometric sequence.

The sum of the first n terms of a geometric sequence:

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1}$$

So

If $r = 1$**, then** $S_n = a_1 + a_1 + a_1 + \cdots + a_1 = na_1$**.**

Otherwise

$$\begin{aligned} S_n &= a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1} \\ -rS_n &= \qquad a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1} + a_1 r^n \\ \hline (1-r)S_n &= a_1 - a_1 r^n \end{aligned}$$

So $S_n = \begin{cases} na_1 & ; r = 1 \\ a_1 \cdot \dfrac{1-r^n}{1-r} & ; r \neq 1 \end{cases}$.

Find the following sums:

1. $\frac{1}{4} + \frac{2}{4} + \frac{2^2}{4} + \cdots + \frac{2^{n-1}}{4}$ **What's *r*? How many terms? What's a_1?**

$$= \frac{1}{4} \cdot 2^{1-1} + \frac{1}{4} \cdot 2^{2-1} + \frac{1}{4} \cdot 2^{3-1} + \cdots + \frac{1}{4} \cdot 2^{n-1}$$

$$S_n = \underbrace{\frac{1}{4}}_{a_1} \cdot \frac{1-2^n}{1-2} =$$

2. $\frac{3}{9} + \frac{3^2}{9} + \frac{3^3}{9} + \cdots + \frac{3^{n+1}}{9}$ **What's *r*? How many terms? What's a_1?**

$$S_{n+1} = \underbrace{\frac{3}{9}}_{a_1} \cdot \frac{1-3^{n+1}}{1-3} =$$

3. $\sum_{k=1}^{n-1}\left(\frac{2}{3}\right)^{k}$ **What's *r*?** **How many terms?** **What's a_1?**

$$S_{n-1} = \underset{a_1}{\frac{2}{3}} \cdot \frac{1-\left(\frac{2}{3}\right)^{n-1}}{1-\frac{2}{3}} =$$

4. $\frac{1}{4}+\frac{2}{4}+\frac{2^2}{4}+\cdots+\frac{2^{14}}{4}$ **What's *r*?** **How many terms?** **What's a_1?**

$$S_{15} = \underset{a_1}{\frac{1}{4}} \cdot \frac{1-2^{15}}{1-2} =$$

5. $\sum_{n=0}^{8}\left(-\frac{2}{3}\right)^n$ **What's r? How many terms? What's a_1?**

$$S_9 = \underset{a_1}{1} \cdot \frac{1-\left(-\frac{2}{3}\right)^9}{1-\left(-\frac{2}{3}\right)} =$$

What if you try to add all the infinitely many terms of a geometric sequence?

Let's see what happens to the sum of the first *n* terms, called the *n*th partial sum, as the number of included terms gets larger. If it settles on a value we'll say that the partial sums converge to that value. Otherwise, we'll say that the partial sums diverge.

If $r = 1$**,** $S_n = na_1$**, and as *n* gets larger and larger, the *n*th partial sum,** S_n**, will head toward** ∞**, if** $a_1 > 0$**, or toward** $-\infty$**, if** $a_1 < 0$**. In this case, the partial sums diverge.**

For $a_1 = 5$**, let's look at the partial sums:**

$S_1 = 5$	$S_2 = 5+5$	$S_3 = 5+5+5$	$S_4 = 5+5+5+5$	$S_5 = 5+5+5+5+5$	$\cdots$
5	**10**	**15**	**20**	**25**	$\cdots$

For $a_1 = -10$, let's look at the partial sums:

$S_1 = -10$	$S_2 = -10 + -10$	$S_3 = -10 + -10 + -10$	$S_4 = -10 + -10 + -10 + -10$	$\cdots$
-10	-20	-30	-40	$\cdots$

If $r = -1$, $S_1 = a_1$, $S_2 = a_1 - a_1 = 0$, $S_3 = a_1 - a_1 + a_1 = a_1$,…, so the partial sums will continue to bounce between a_1 and zero as *n* continues to increase. In this case, the partial sums diverge.

For $a_1 = 1$, let's look at the partial sums:

$S_1 = 1$	$S_2 = 1 + -1$	$S_3 = 1 + -1 + 1$	$S_4 = 1 + -1 + 1 + -1$	$S_5 = 1 + -1 + 1 + -1 + 1$
1	**0**	**1**	**0**	**1**

If $r = 0$, $S_n = a_1 + 0 + 0 + \cdots + 0 = a_1$, so the partial sums always have the value a_1 as *n* continues to increase. In this case, we say that the partial sums converge to a_1.

For $a_1 = -2$, let's look at the partial sums:

$S_1 = -2$	$S_2 = -2 + 0$	$S_3 = -2 + 0 + 0$	$S_4 = -2 + 0 + 0 + 0$	$S_5 = -2 + 0 + 0 + 0 + 0$
-2	-2	-2	-2	-2

Since, for $r \neq 1$ $S_n = a_1 \cdot \dfrac{1 - r^n}{1 - r}$, the only chance for it to settle is if r^n settles. This only happens when $-1 < r < 1$, in which case, r^n settles on zero.

So $a_1 + a_1 r + a_1 r^2 + \cdots = \sum_{n=1}^{\infty} a_1 r^{n-1}$ converges to $a_1 \cdot \dfrac{1}{1 - r}$ if $-1 < r < 1$, and diverges otherwise. The attempt at adding up all the terms of a geometric sequence, $a_1 + a_1 r + a_1 r^2 + \cdots = \sum_{n=1}^{\infty} a_1 r^{n-1}$, is called an infinite geometric series, or just a geometric series.

In short, a geometric series converges to $\dfrac{\text{first term}}{1-r}$ **if** $|r|<1$**, and diverges if** $|r|\geq 1$**.**

<u>Examples:</u>

Determine if the following geometric series are convergent or divergent. If convergent, tell what they converge to.

1. $2+\dfrac{4}{3}+\dfrac{8}{9}+\cdots$

S_1	S_2	S_3	S_4	S_5	S_6	S_7
2	$\dfrac{10}{3}=3.\overline{3}$	$\dfrac{38}{9}=4.\overline{2}$	$\dfrac{130}{27}=4.\overline{814}$	$\dfrac{422}{81}\approx 5.21$	$\dfrac{1330}{243}\approx 5.47$	$\dfrac{4118}{729}\approx 5.65$
S_8	S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}
≈ 5.77	≈ 5.84	≈ 5.90	≈ 5.93	≈ 5.95	≈ 5.97	≈ 5.98

2. $8+4+2+\cdots$

S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8
8	12	14	15	$\frac{31}{2}=15.5$	$\frac{63}{4}=15.75$	$\frac{127}{8}=15.875$	$\frac{255}{16}=15.9375$
S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}	S_{15}	S_{16}
≈ 15.969	≈ 15.984	≈ 15.992	≈ 15.996	≈ 15.998	≈ 15.999	≈ 16.000	≈ 16.000

3. $2 - \frac{1}{2} + \frac{1}{8} - \frac{1}{32} + \cdots$

S_1	S_2	S_3	S_4	S_5	S_6
2	$\frac{3}{2} = 1.5$	$\frac{13}{8} = 1.625$	$\frac{51}{32} = 1.59375$	$\frac{205}{128} = 1.6015625$	$\frac{819}{512} = 1.599609375$
S_7	S_8	S_9	S_{10}	S_{11}	S_{12}
≈ 1.6000061	≈ 1.5999985	≈ 1.6000004	≈ 1.5999999	≈ 1.6000000	≈ 1.6000000

4. $\sum_{k=1}^{\infty} 3 \cdot \left(\frac{3}{2}\right)^{k-1}$

S_1	S_2	S_3	S_4	S_5	S_6
3	$\frac{15}{2} = 7.5$	$\frac{57}{4} = 14.25$	$\frac{195}{8} = 24.375$	$\frac{633}{16} = 39.5625$	$\frac{1995}{32} = 62.34375$
S_7	S_8	S_9	S_{10}	S_{11}	S_{12}
≈ 96.52	≈ 147.77	≈ 224.66	≈ 339.99	≈ 512.99	≈ 772.48

5. $\sum_{n=0}^{\infty} 3 \cdot \left(\frac{2}{3}\right)^n$

S_1	S_2	S_3	S_4	S_5	S_6	S_7
3	5	$\frac{19}{3} = 6.\overline{3}$	$\frac{65}{9} = 7.\overline{2}$	$\frac{211}{27} = 7.\overline{814}$	$\frac{665}{81} \approx 8.21$	$\frac{2059}{243} \approx 8.47$
S_8	S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}
≈ 8.65	≈ 8.77	≈ 8.84	≈ 8.90	≈ 8.93	≈ 8.95	≈ 8.97

6. $\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n$

S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8
1	$\frac{1}{2}=.5$	$\frac{3}{4}=.75$	$\frac{5}{8}=.625$	$\frac{11}{16}=.6875$	$\frac{21}{32}=.65625$	$\frac{43}{64}=.671875$	$\frac{85}{128}=.6640625$
S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}	S_{15}	S_{16}
$\approx .66797$	$\approx .66602$	$\approx .66699$	$\approx .66650$	$\approx .66675$	$\approx .66663$	$\approx .66669$	$\approx .66666$

<u>Other Infinite Series:</u>

In general, all infinite series, $a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$**, can be analyzed by looking at their related sequence of partial sums.**

Examples:

1. $\sum_{n=1}^{\infty} n$

It's sequence of partial sums is given by $S_n = 1 + 2 + 3 + \cdots + n = \frac{1}{2} n(n+1)$**. Clearly, as** $n \to \infty$**, the values of the partial sums go to** ∞**, so this series is divergent.**

2. $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$

It's sequence of partial sums is given by $S_n = \left(1-\tfrac{1}{2}\right)+\left(\tfrac{1}{2}-\tfrac{1}{3}\right)+\left(\tfrac{1}{3}-\tfrac{1}{4}\right)+\cdots+\left(\tfrac{1}{n}-\tfrac{1}{n+1}\right)$**.**

$$= 1-\tfrac{1}{n+1}$$

As $n\to\infty$**, the values of the partial sums go to 1, so this series converges to 1.**

3. $\sum_{n=1}^{\infty}\ln\left(\frac{n}{n+1}\right)$

It's sequence of partial sums is given by

$$\begin{aligned} S_n &= \ln\tfrac{1}{2}+\ln\tfrac{2}{3}+\ln\tfrac{3}{4}+\cdots+\ln\tfrac{n}{n+1} \\ &= (\ln 1-\ln 2)+(\ln 2-\ln 3)+(\ln 3-\ln 4)+\cdots+\left[\ln n-\ln(n+1)\right] \\ &= \ln 1-\ln(n+1) \\ &= -\ln(n+1) \end{aligned}$$

.

As $n\to\infty$**, the values of the partial sums go to** $-\infty$**, so this series diverges.**