<u>Proving More Things Using the Principle of Mathematical Induction:</u>

Divisibility by 3: A whole number, *n*, is divisible by 3, if there is a whole *k* so that $n = 3k$.

Must the sum of two whole numbers that are divisible by 3 also be divisible by 3?

Must the product of a whole number and a whole number divisible by 3 be divisible by 3?

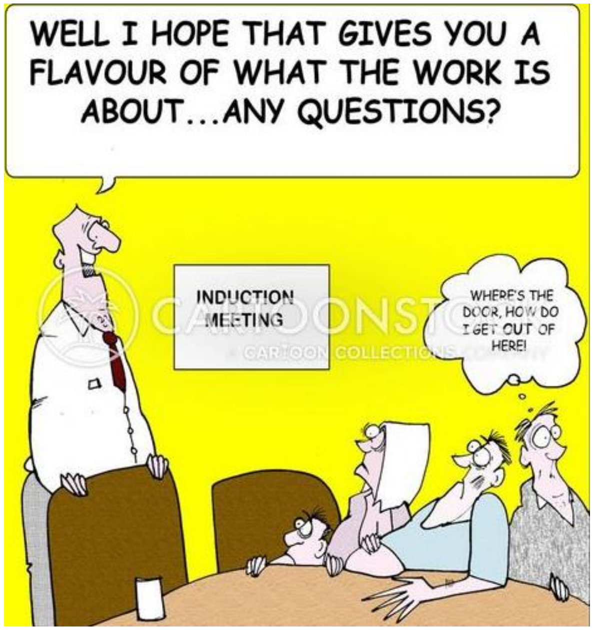
1. Prove that $4^n - 1$ is divisible by 3 for all natural numbers, *n*.

<u>Base Step:</u> Show that it's true for $n = 1$. $4^1 - 1 =$

<u>Induction Step:</u> Suppose that it's true for $n = k$. So $4^k - 1$ is divisible by 3.

<u>Goal:</u> Show that it's true for $n = k + 1$, i.e. $4^{k+1} - 1$ is divisible by 3.

$$4^{k+1} - 1 = \underbrace{4\left(4^k - 1\right)} + \quad 3$$



<u>Conclusion:</u>

2. Prove that $n^3 + 2n$ is divisible by 3 for all natural numbers, *n*.

<u>Base Step:</u> Show that it's true for $n = 1$. $1^3 + 2 \cdot 1 =$

<u>Induction Step:</u> Suppose that it's true for $n = k$. So $k^3 + 2k$ is divisible by 3.

<u>Goal:</u> Show that it's true for $n = k+1$, i.e. $(k+1)^3 + 2(k+1)$ is divisible by 3.

$$
\begin{aligned}
(k+1)^3 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 2k + 2 \\
&= \left(k^3 + 2k\right) + \left(3k^2 + 3k + 3\right) \\
&= \underbrace{\left(k^3 + 2k\right)} + \underbrace{3\left(k^2 + k + 1\right)}
\end{aligned}
$$

<u>Conclusion:</u>

Sometimes statements involving natural numbers aren't true for all natural numbers.

Sometimes they're only true for natural numbers *n* with $n \geq n_0$**.**

1. Prove that $\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$ **for all natural numbers, *n*, with** $n \geq 2$**.**

<u>Base Step:</u> Show that it's true for $n=2$**.**

Left side	**Right side**
$1-\frac{1}{4}=$	$\frac{2+1}{2\cdot 2}=$

Induction Step: **Suppose that it's true for $n=k$. So**

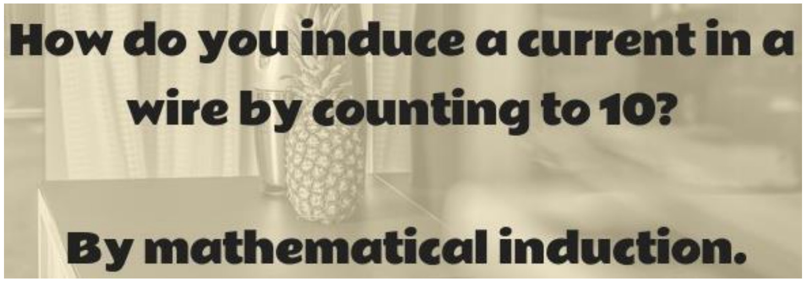
$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{k^2}\right)=\frac{k+1}{2k}.$$

Multiply both sides by $\left[1-\frac{1}{(k+1)^2}\right]$ **to get**

$$\underbrace{\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{k^2}\right)\left[1-\frac{1}{(k+1)^2}\right]}_{n=k+1}=\frac{k+1}{2k}\left[1-\frac{1}{(k+1)^2}\right]$$

$$=\frac{k+1}{2k}\left[\frac{(k+1)^2}{(k+1)^2}-\frac{1}{(k+1)^2}\right]=\frac{k+1}{2k}\left[\frac{k^2+2k}{(k+1)^2}\right]=\frac{k+1}{2k}\left[\frac{k(k+2)}{(k+1)^2}\right]=$$

Conclusion:



2. Prove that $n^2 > n+1$ for all natural numbers, *n*, with $n \geq 2$.

<u>Base Step:</u> Show that it's true for $n = 2$.

Left side	**Right side**
$2^2 =$	$2+1=$

<u>Induction Step:</u> Suppose that it's true for $n = k$. So $k^2 > k+1$.

$$(k+1)^2 = k^2 + 2k + 1 > (k+1) + 1 + 2k > (k+1) +$$

<u>Conclusion:</u>

Factorials:

$$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$$
$$= n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$$

$$1! = 1$$

$$2! = 2 \cdot 1 = 2$$

$$3! = 3 \cdot 2 \cdot 1 = 6$$

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 4 \cdot 3! = 4 \cdot 6 = 24$$

$$5! = 5 \cdot 4! = 120$$

$$6! = 6 \cdot 5! = 720$$

By special separate definition, $0! = 1$**.**

3. Prove that $n! > n^2$ for all natural numbers, *n*, with $n \geq 4$.

<u>Base Step:</u> Show that it's true for $n = 4$.

Left side	**Right side**
$4! =$	$4^2 =$

<u>Induction Step:</u> Suppose that it's true for $n = k$. So $k! > k^2$. Multiply both sides of the inequality by $(k+1)$, to get

$$(k+1)! > \underbrace{k^2(k+1) > (k+1)(k+1)}_{\text{from the previous induction problem}}$$

<u>Conclusion:</u>

Sometimes statements that can be proven by induction can also be proven in another way.

1. Prove that $n^3 + 3n^2 + 2n$ **is divisible by 3 for all natural numbers, *n*.**

$n^3 + 3n^2 + 2n = n(n^2 + 3n + 2) = n(n+1)(n+2)$. **Every third natural number starting with 1 is a multiple of 3, so if you have three consecutive natural numbers, one of them must be a multiple of 3, and hence their product is a multiple of 3.**

2. Prove that $n^3 + 3n^2 + 2n$ **is divisible by 6 for all natural numbers, *n*.**

$n^3 + 3n^2 + 2n = n(n^2 + 3n + 2) = n(n+1)(n+2)$. **Every second natural number starting with 1 is a multiple of 2, so if you have three consecutive natural numbers, at least one of them must be a multiple of 2, and hence their product is a multiple of 2 and 3. This means their product must be a multiple of 6.**

3. Prove that $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\cdots\left(1-\frac{1}{n}\right)=\frac{1}{n}$ **for all natural numbers, *n*, with** $n \geq 2$**.**

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\cdots\left(1-\frac{1}{n}\right)=\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdots\cdots\frac{n-2}{n-1}\cdot\frac{n-1}{n}=$$