Binomial Expansions:

Multiplying out a power of a binomial is called expanding the binomial (power).

$$(x+a)^0 = 1$$

$$(x+a)^1 = x+a$$

$$(x+a)^2 = x^2 + 2ax + a^2$$

$$(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$$

If we just look at the coefficients of the terms in the expansions so far, here's what we get:

$(x+a)^0$**:**				**1**			
$(x+a)^1$**:**			**1**		**1**		
$(x+a)^2$**:**		**1**		**2**		**1**	
$(x+a)^3$**:**	**1**		**3**		**3**		**1**

This is the beginning of a triangle of coefficients called Pascal's Triangle.

There is a nice additive pattern to the coefficients.

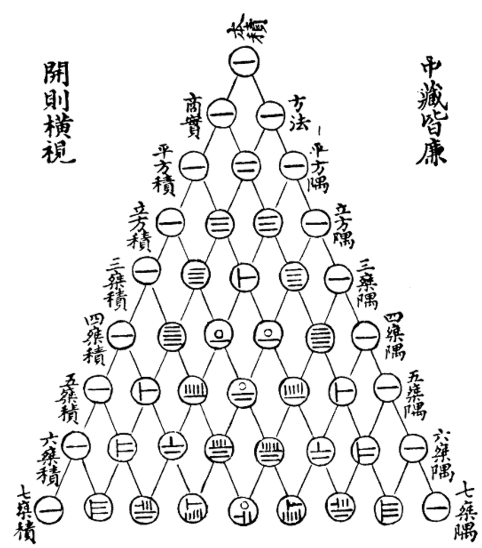
Let's see why it's true in going from $(x+a)^2 = 1x^2 + 2ax + 1a^2$ to $(x+a)^3$.

$$(x+a)^3 = (x+a)(x+a)^2 = (x+a)(x^2+2ax+a^2)$$
$$= x^3 + 2ax^2 + a^2x + ax^2 + 2a^2x + a^3$$
$$= 1x^3 + (1+2)ax^2 + (2+1)a^2x + 1a^3$$

1 2 1

1 1

1



Let's continue Pascal's Triangle a little further.

$(x+a)^0$:						**1**					
$(x+a)^1$:					**1**		**1**				
$(x+a)^2$:				**1**		**2**		**1**			
$(x+a)^3$:			**1**		**3**		**3**		**1**		
$(x+a)^4$:		**1**		**4**		**6**		**4**		**1**	
$(x+a)^5$:	**1**		**5**		**10**		**10**		**5**		**1**

If you look at the powers of *x* and *a* in the terms, there is another pattern. The powers of *x* decrease by 1 from left to right, while the powers of *a* increase by 1 from left to right. Also the sum of the powers is always equal to the exponent on the binomial.

Let's use Pascal's Triangle to complete the expansion of the binomial $(x+1)^4$.

$$(x+1)^4 = \square x^4 \cdot 1^0 + \square x^3 \cdot 1^1 + \square x^2 \cdot 1^2 + \square x^1 \cdot 1^3 + \square x^0 \cdot 1^4$$

$$=$$

Let's use Pascal's Triangle to complete the expansion of the binomial $(x+2)^3$.

$$(x+2)^3 = \square x^3 \cdot 2^0 + \square x^2 \cdot 2^1 + \square x^1 \cdot 2^2 + \square x^0 \cdot 2^3$$

$$=$$

Let's use Pascal's Triangle to complete the expansion of the binomial $(x-1)^5$.

$$(x-1)^5 = \square x^5 \cdot (-1)^0 + \square x^4 \cdot (-1)^1 + \square x^3 \cdot (-1)^2 + \square x^2 \cdot (-1)^3 + \square x^1 \cdot (-1)^4 + \square x^0 \cdot (-1)^5$$

$$=$$

Let's use Pascal's Triangle to complete the expansion of the binomial $(3x-2)^4$.

$$(3x-2)^4 = \square (3x)^4 \cdot (-2)^0 + \square (3x)^3 \cdot (-2)^1 + \square (3x)^2 \cdot (-2)^2 + \square (3x)^1 \cdot (-2)^3 + \square (3x)^0 \cdot (-2)^4$$

$$=$$

Pascal's Triangle is a nice way to generate the coefficients in a binomial expansion, but unfortunately, it finds them recursively. There is a direct way of finding the coefficients using a formula that involves factorials.

$$\begin{aligned} n! &= 1\cdot 2\cdot 3\cdot\cdots\cdot n \\ &= n\cdot(n-1)\cdot(n-2)\cdot\cdots\cdot 2\cdot 1 \end{aligned}$$

$$1! = 1$$

$$2! = 2\cdot 1 = 2$$

$$3! = 3\cdot 2\cdot 1 = 6$$

$$4! = 4\cdot 3\cdot 2\cdot 1 = 4\cdot 3! = 4\cdot 6 = 24$$

$$5! = 5\cdot 4! = 120$$

$$6! = 6\cdot 5! = 720$$

By special separate definition, $0! = 1$**.**

<u>Binomial Coefficient Formula:</u>

$$\binom{n}{j} = \frac{n!}{j! \cdot (n-j)!} \quad \text{on calculators: } {}_nC_j$$

1. $\binom{5}{0}$

$\left\{\binom{n}{0}\right\}$

2. $\binom{5}{1}$

$\left\{\binom{n}{1}\right\}$

3. $\binom{5}{2}$

4. $\binom{5}{3}$

5. $\binom{5}{4}$

$\left\{\binom{n}{n-1}\right\}$

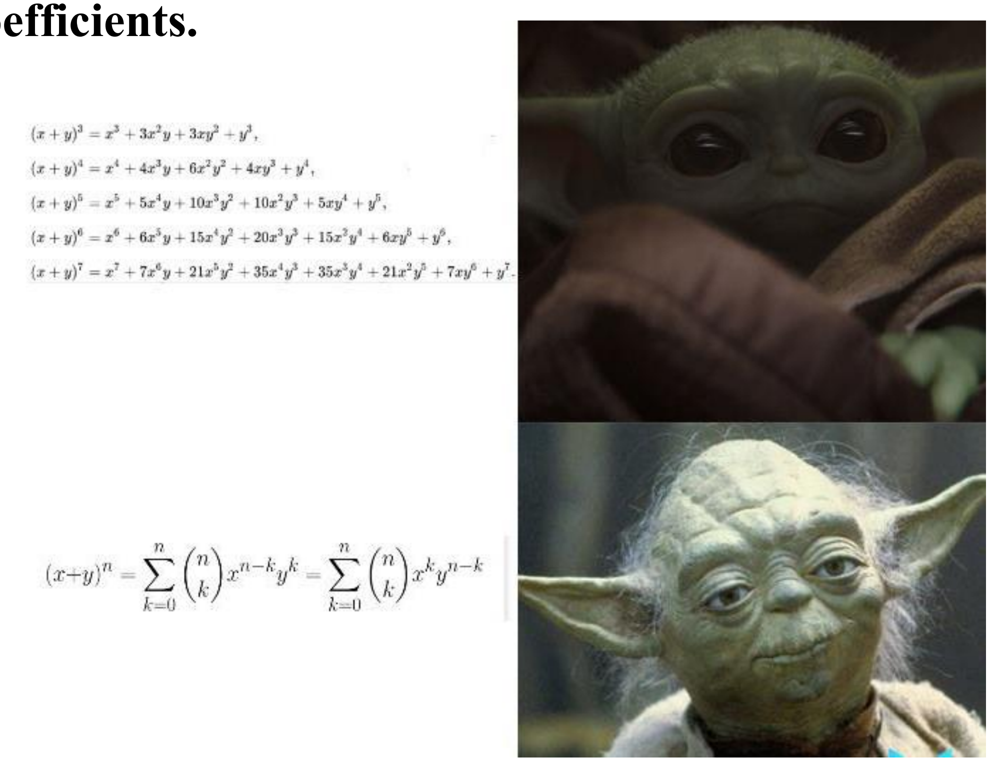
6. $\binom{5}{5}$

$\left\{\binom{n}{n}\right\}$

<u>Binomial Theorem:</u>

$$
\begin{aligned}
(x+a)^n &= \binom{n}{0}x^n a^0 + \binom{n}{1}x^{n-1}a^1 + \binom{n}{2}x^{n-2}a^2 + \cdots + \binom{n}{n-1}x^1 a^{n-1} + \binom{n}{n}x^0 a^n \\
&= \sum_{j=0}^{n} \binom{n}{j} x^{n-j} a^j
\end{aligned}
$$

When using the Binomial Theorem, you may use the binomial coefficient formula or Pascal’s Triangle to find the values of the coefficients.



$$(x+1)^6 = \binom{6}{0}x^6 + \binom{6}{1}x^5 + \binom{6}{2}x^4 + \binom{6}{3}x^3 + \binom{6}{4}x^2 + \binom{6}{5}x + \binom{6}{6}$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Examples:

1. Find the coefficient of x^3 **in the expansion of** $(x+3)^{10}$**.**

2. Find the coefficient of x^2 **in the expansion of** $(2x-3)^9$**.**

3. Find the sixth term in the expansion of $(3x+2)^8$.

4. Find the coefficient of x^0 in the expansion of $\left(x-\frac{1}{x^2}\right)^9$.

5. Find the coefficient of x^2 in the expansion of $\left(\sqrt{x}+\frac{3}{\sqrt{x}}\right)^8$.

6. Find the exact value of $\binom{1{,}000}{0}+\binom{1{,}000}{1}+\binom{1{,}000}{2}+\cdots+\binom{1{,}000}{1{,}000}$.

$$\left\{\sum_{j=0}^{1{,}000}\binom{1{,}000}{j}\square^{1000-j}\square^{j}=(\square+\square)^{1{,}000}\right\}$$

7. Find the exact value of $\binom{1{,}000}{0}-\binom{1{,}000}{1}+\binom{1{,}000}{2}-\binom{1{,}000}{3}+\cdots+\binom{1{,}000}{1{,}000}$.

$$\left\{\sum_{j=0}^{1{,}000}\binom{1{,}000}{j}\square^{1000-j}\square^{j}=\left(\square+\square\right)^{1{,}000}\right\}$$

BINOMIAL EXPANSION:

FINDING A TERM OF COEFFICIENT

Find the third term of $(2+3x)^5$