Dot Product of Two Vectors:

If $\vec{v} = \langle a_1, b_1 \rangle$ **and** $\vec{w} = \langle a_2, b_2 \rangle$**, then the dot product of the two vectors is** $\vec{v} \cdot \vec{w} = a_1 a_2 + b_1 b_2$

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Examples:

1. $\langle 1,2 \rangle \cdot \langle 2,-3 \rangle$

2. $\langle 1,2 \rangle \cdot \langle 2,-1 \rangle$

3. $\langle 1,2 \rangle \cdot \langle 1,2 \rangle$

4. $\langle 1,2 \rangle \cdot \langle 0,0 \rangle$

<u>Properties of Dot Product:</u>

1. $\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}$

Prove it.

2. $\vec{u}\cdot\left(\vec{v}+\vec{w}\right)=\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$

Prove it.

3. $\vec{u}\cdot\vec{u}=\left\|\vec{u}\right\|^{2}$

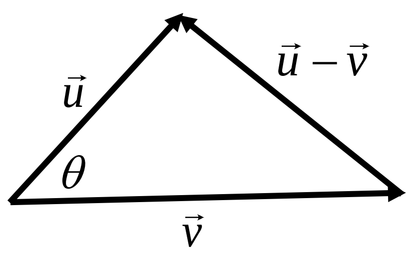
Prove it.

4. $\vec{u}\cdot\vec{0}=\vec{0}\cdot\vec{u}=0$

Prove it.

What does the dot product represent?

If $\vec{u}$ and $\vec{v}$ are non-zero vectors, then we can form the following triangle.



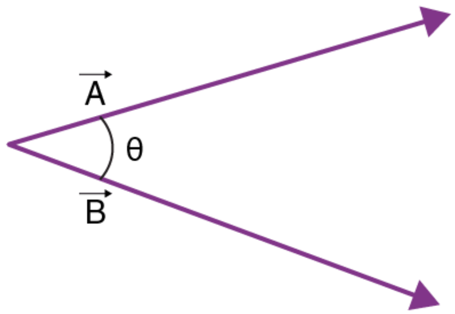
From the Law of Cosines, we get that

$$\|\vec{u}-\vec{v}\|^2=\|\vec{u}\|^2+\|\vec{v}\|^2-2\|\vec{u}\|\|\vec{v}\|cos\,\theta$$

So

$$(\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})=\vec{u}\cdot\vec{u}+\vec{v}\cdot\vec{v}-2\|\vec{u}\|\|\vec{v}\|cos\,\theta$$

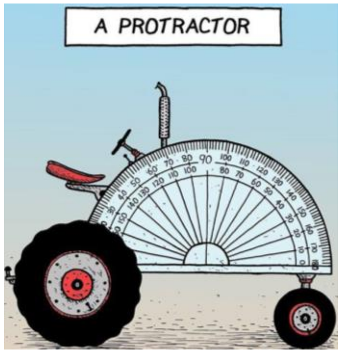
For $\vec{u}$ and $\vec{v}$ non-zero vectors, we can find the angle between the vectors, θ, with $0 \le \theta \le \pi$, from the formula



$$cos\,\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|}.$$

Example:

Find the angle between the vectors $\langle \sqrt{3}, 1 \rangle$ and $\langle -2\sqrt{3}, 2 \rangle$.



<u>Parallel Vectors:</u>

Two non-zero vectors $\vec{u}$ and $\vec{v}$ are parallel if one of them is a scalar multiple of the other, i.e. $\vec{u} = \alpha\vec{v}$. If $\alpha > 0$, then the angle between the vectors is 0. If $\alpha < 0$, then the angle between the vectors is π. The zero vector is considered to be parallel to all vectors.

Orthogonal Vectors:

Two non-zero vectors $\vec{u}$ and $\vec{v}$ are orthogonal(perpendicular) if the angle between them is $\frac{\pi}{2}$. In this case, $\vec{u}\cdot\vec{v}=0$. The zero vector is considered to be orthogonal to every vector, so two vectors $\vec{u}$ and $\vec{v}$ are orthogonal if and only if $\vec{u}\cdot\vec{v}=0$.

Examples:

Determine if the given pair of vectors are parallel, orthogonal, or neither.

1. $\langle 1,2\rangle$ **and** $\langle 3,6\rangle$

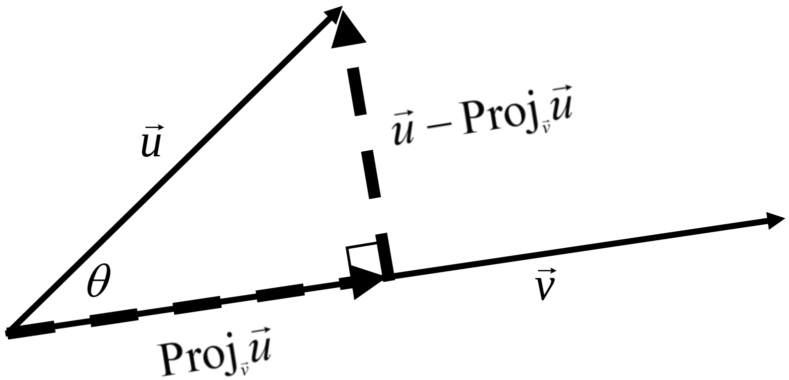
2. $\langle 1,2\rangle$ **and** $\left\langle -\frac{1}{3},-\frac{2}{3}\right\rangle$

3. $\langle 1,2\rangle$ **and** $\langle -2,1\rangle$

4. $\langle 1,2\rangle$ **and** $\langle 4,-7\rangle$

Vector Projection:

Given two non-zero vectors $\vec{u}$ and $\vec{v}$, we would like to write the vector $\vec{u}$ as a sum of two vectors: one parallel to $\vec{v}$ and one orthogonal to $\vec{v}$.



$$\vec{u} = \text{Proj}_{\vec{v}}\vec{u} + \left(\vec{u} - \text{Proj}_{\vec{v}}\vec{u}\right)$$

$$\text{Proj}_{\vec{v}}\vec{u} = \|\vec{u}\| cos\,\theta \frac{\vec{v}}{\|\vec{v}\|} = \left(\frac{\|\vec{u}\|\|\vec{v}\| cos\,\theta}{\|\vec{v}\|^2}\right)\vec{v} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$$

Example:

Write the vector $\vec{u} = \langle 2,3 \rangle$ as a sum of two vectors: one parallel to $\vec{v} = \langle 1,-3 \rangle$ and one orthogonal to $\vec{v} = \langle 1,-3 \rangle$.