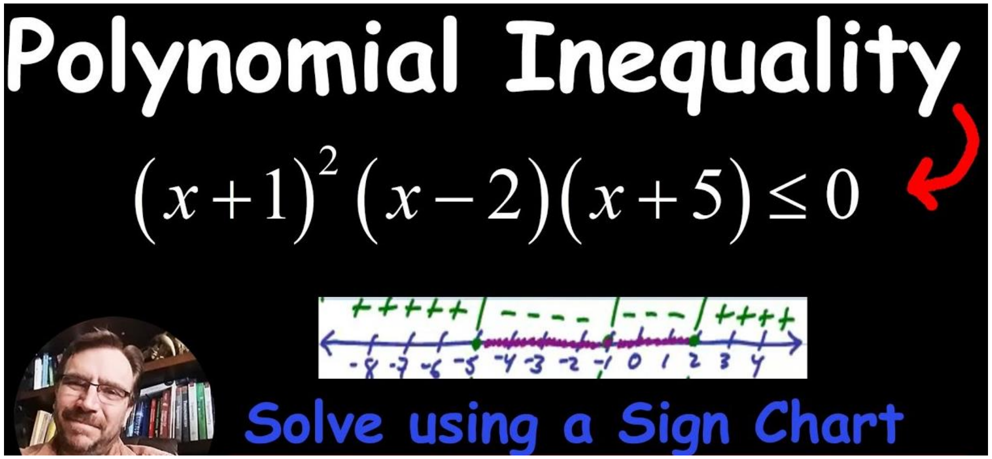
Review of Polynomial and Rational Inequalities:

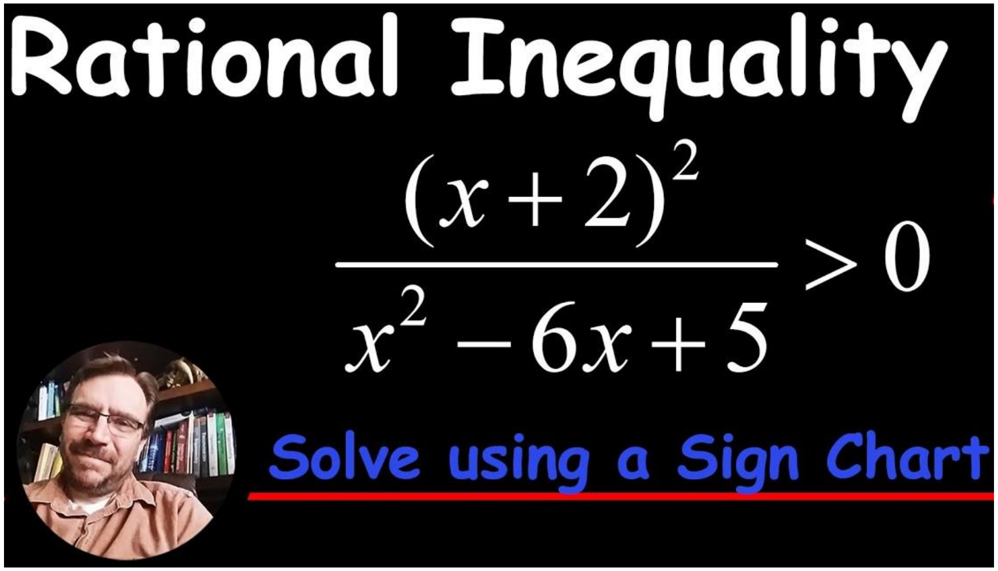
To solve a polynomial or rational inequality, just do the following steps:

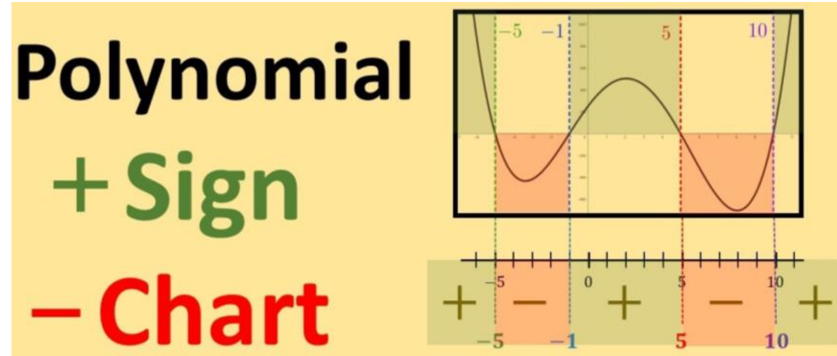
1. Get zero on one side.

2. Create the sign chart for the other side. ***(If factors can be cancelled out, then do so.)***

3. Read the solution from the sign chart.

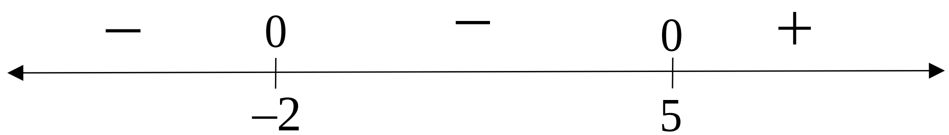






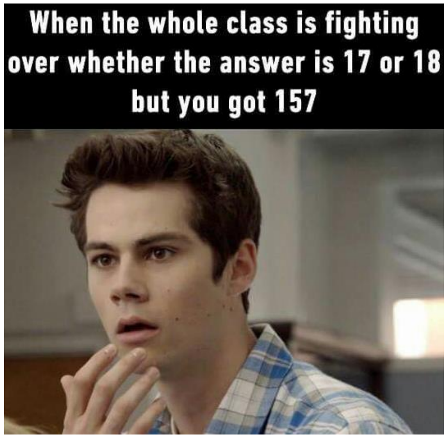
Examples:

1. $(x-5)(x+2)^2 > 0$



The solution in interval notation is $(5,\infty)$**.**

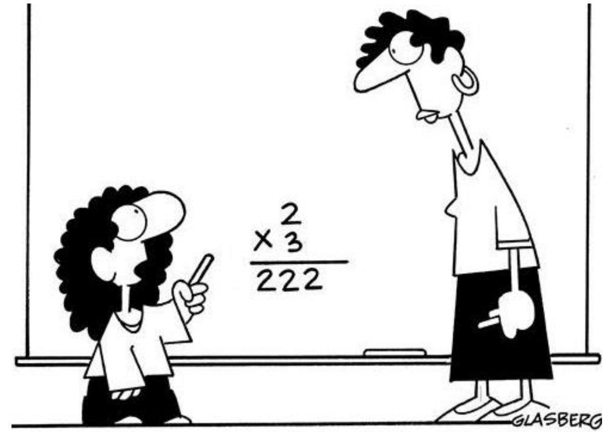
2. $x^3 + 8x^2 < 0$



3. $x^3 + 2x^2 - 3x \geq 0$

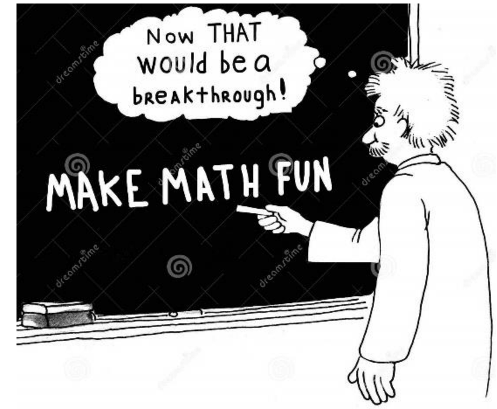
4. $x^4 \leq 9x^2$

5. $x^2 + 4 \le 4x$



"What do you mean, it's the wrong kind of right?"

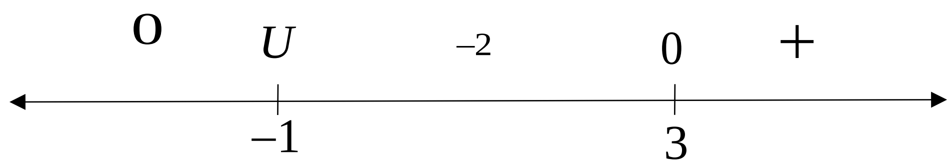
6. $x^2 - 4x \le -2$

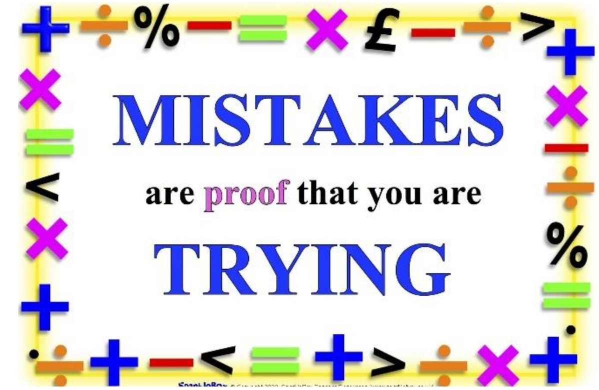


7. $x^2 - 4x + 5 < 0$

8. What's the domain of the function $f\left(x\right) = \sqrt{x^4 - 3x^2 + 2}$**?**

9. $\dfrac{x-3}{x+1} \geq 0$





The solution in interval notation is $(-\infty,-1)\cup[3,\infty)$**.**

10. $\dfrac{(x-2)^2}{x^2-1} \geq 0$

11. $\dfrac{x+4}{x-2} \le 1$

12. $\dfrac{5}{x-3} > \dfrac{3}{x+1}$

13. $\dfrac{1}{x-2} < \dfrac{2}{3x-9}$

14. What's the domain of the function $f(x) = \sqrt{\dfrac{(x-1)^2}{(x+2)(x-3)}}$ **?**

Partial Fraction Decompositions:

Algebraic fractions can be added by finding a common denominator, modifying the original numerators, and then combining them.

$$\frac{1}{x-1}+\frac{2}{x+2}=\frac{x+2}{(x-1)(x+2)}+\frac{2(x-1)}{(x-1)(x+2)}$$
$$=\frac{x+2}{(x-1)(x+2)}+\frac{2x-2}{(x-1)(x+2)}$$
$$=\frac{3x}{(x-1)(x+2)}$$

What is Beethoven doing now?

De-composing.

A partial fraction decomposition is a reversal of this process.

Start with an algebraic fraction like $\frac{3x}{(x-1)(x+2)}$**, and determine the algebraic fractions whose sum would be** $\frac{3x}{(x-1)(x+2)}$**. In this case it's** $\frac{1}{x-1}+\frac{2}{x+2}$**.**

In general, you'll start with a rational function, $\frac{p(x)}{q(x)}$, and you'll try to write it as

$$\frac{p(x)}{q(x)} = \underbrace{f_1(x) + f_2(x) + \cdots + f_r(x)}_{\text{Partial Fraction Decomposition}}.$$

The process begins by forming a guess for the decomposition that contains parameters(variables) whose values are to be determined.

The values of the parameters can be determined by plugging in values of *x*, matching up coefficients, or a combination of the two.

When the values of the parameters are plugged into the guess, the partial fraction decomposition will be complete.

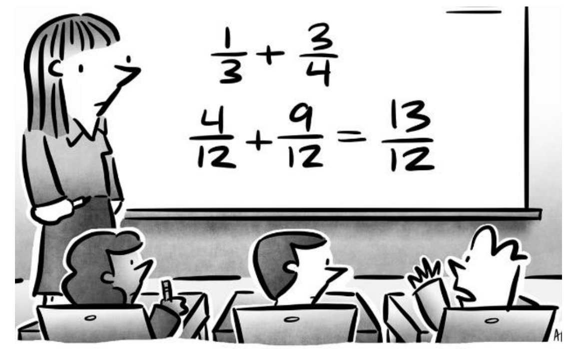
Assuming that the degree of $q(x)$ is greater than the degree of $p(x)$,

1. Factor $q(x)$.

2. If $(ax+b)^k$ is the highest power of $(ax+b)$ which is a factor of , $q(x)$ then your guess must include the terms $\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k}$.

3. If $(ax^2+bx+c)^k$ is the highest power of (ax^2+bx+c) which is a factor of , $q(x)$ then your guess must include the terms

$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$**, where** ax^2+bx+c **is an irreducible quadratic factor, i.e. it doesn't have real zeros.**



"OK, we made it so they had something in common, added them together, and then the end result is *improper*? I mean, I kinda feel like we just made things worse!

<u>Examples:</u>

1. $\dfrac{3x}{(x+2)(x-4)}$

$\dfrac{3x}{(x+2)(x-4)} = \dfrac{A}{x+2} + \dfrac{B}{x-4}$**, so multiply both sides by the denominator on the left to get**

$$3x = A(x-4) + B(x+2)$$

Method#1: Expand the right side, collect like terms, and equate coefficients and constants on both sides.

$$3x = Ax - 4A + Bx + 2B \Rightarrow 3x + 0 = (A+B)x + (2B-4A), \text{ so } \begin{matrix} A+B=3 \\ 2B-4A=0 \end{matrix}.$$

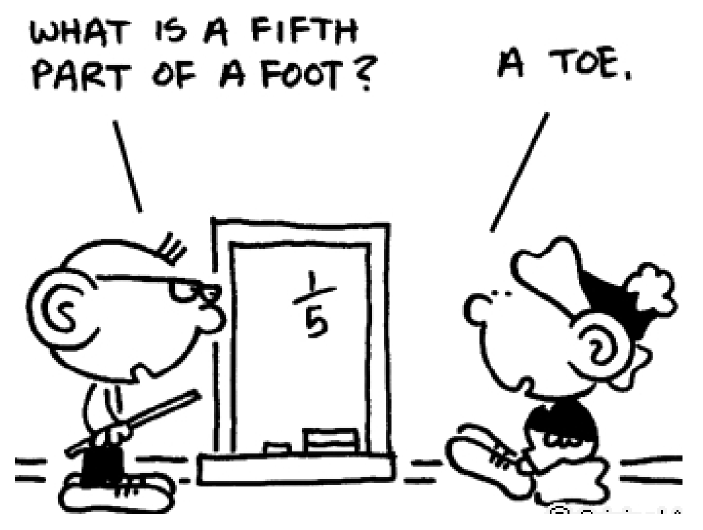
Solving the system leads to $A=1, B=2$**.**

Method#2: Pick numerical values for *x* that make it easy to determine the values of *A* and *B*.

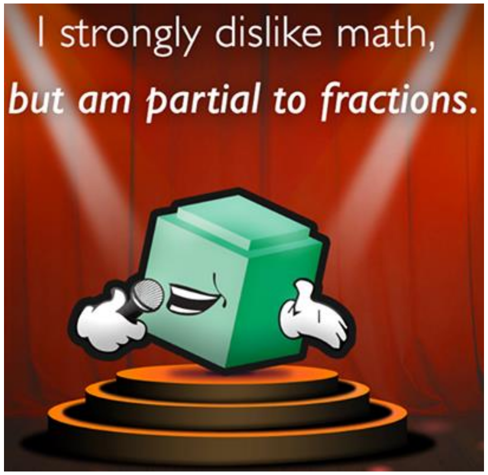
Let $x = 4$ **to get the equation** $12 = 6B \Rightarrow B = 2$**, and let** $x = -2$ **to get the equation** $-6 = -6A \Rightarrow A = 1$**.**

Either method leads to a partial fraction decomposition of

$$\frac{3x}{(x+2)(x-4)} = \frac{1}{x+2} + \frac{2}{x-4}.$$



2. $\dfrac{x}{x^2+2x-3}$ $\qquad\left\{x^2+2x-3=(x+3)(x-1)\right\}$



$$\frac{x}{(x+3)(x-1)}=\frac{A}{x+3}+\frac{B}{x-1}\Rightarrow x=A(x-1)+B(x+3)$$

3. $\dfrac{1}{x\left(x^2+1\right)}$

$$\frac{1}{x\left(x^2+1\right)}=\frac{A}{x}+\frac{Bx+C}{x^2+1}\Rightarrow 1=A\left(x^2+1\right)+x\left(Bx+C\right)$$

4. $\dfrac{x+1}{x^2(x-2)}$

$$\frac{x+1}{x^2(x-2)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-2}\Rightarrow x+1=Ax(x-2)+B(x-2)+Cx^2$$

5. $\dfrac{x+4}{x^2\left(x^2+4\right)}$

$$\frac{x+4}{x^2\left(x^2+4\right)}=\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+4}\Rightarrow x+4=Ax\left(x^2+4\right)+B\left(x^2+4\right)+x^2\left(Cx+D\right)$$

Darling, you complete me!

I don't celebrate Valentine's Day on 2/14.

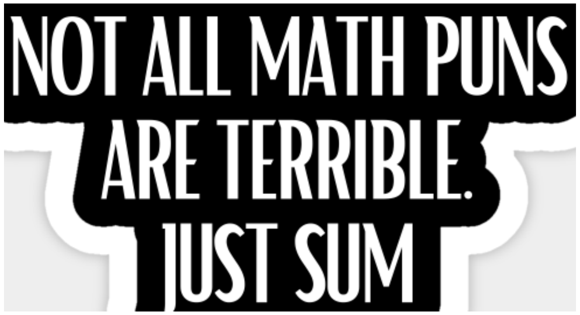
I celebrate on 1/7 because I know how to simplify fractions.

6. $\dfrac{x^2+2x+3}{\left(x^2+4\right)^2}$

$$\frac{x^2+2x+3}{\left(x^2+4\right)^2}=\frac{Ax+B}{x^2+4}+\frac{Cx+D}{\left(x^2+4\right)^2}\Rightarrow x^2+2x+3=\left(Ax+B\right)\left(x^2+4\right)+Cx+D$$

7. $\dfrac{x^3+x^2-3}{x^2+3x-4}$ **{Divide first.}**

$$
\begin{array}{r|l}
 & \quad x \\
\cline{2-2}
x^2+3x-4 & \quad x^3+x^2-3 \\
 & -\left(x^3+3x^2-4x\right) \\
\cline{2-2}
 & -2x^2+4x-3
\end{array}
$$



8. Find the exact value of the following sum of 999,999 terms:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{999{,}999\cdot 1{,}000{,}000}$$

A general term in the sum looks like $\frac{1}{x\left(x+1\right)}$**, so let's do a partial fraction decomposition of it.**

$$\frac{1}{x\left(x+1\right)}=\frac{A}{x}+\frac{B}{x+1}\Rightarrow 1=A\left(x+1\right)+Bx$$

9. Find the exact value of the following sum of 999,998 terms:

$$\frac{2}{1\cdot 2\cdot 3}+\frac{2}{2\cdot 3\cdot 4}+\frac{2}{3\cdot 4\cdot 5}+\cdots+\frac{2}{999{,}998\cdot 999{,}999\cdot 1{,}000{,}000}$$

"You'll find that math gets easier as you learn how to use a calculator."

A general term in the sum looks like $\dfrac{2}{x(x+1)(x+2)}$**, and if you do a partial fraction decomposition, you'll get**

$$\frac{2}{x(x+1)(x+2)}=\frac{1}{x}-\frac{2}{x+1}+\frac{1}{x+2}=\left(\frac{1}{x}-\frac{1}{x+1}\right)-\left(\frac{1}{x+1}-\frac{1}{x+2}\right).$$

So

$$\frac{2}{1\cdot 2\cdot 3}+\frac{2}{2\cdot 3\cdot 4}+\frac{2}{3\cdot 4\cdot 5}+\cdots+\frac{2}{999{,}998\cdot 999{,}999\cdot 1{,}000{,}000}$$

$$=\left[\left(1-\tfrac{1}{2}\right)+\left(\tfrac{1}{2}-\tfrac{1}{3}\right)+\left(\tfrac{1}{3}-\tfrac{1}{4}\right)+\cdots+\left(\tfrac{1}{999{,}998}-\tfrac{1}{999{,}999}\right)\right]-\left[\left(\tfrac{1}{2}-\tfrac{1}{3}\right)+\left(\tfrac{1}{3}-\tfrac{1}{4}\right)+\left(\tfrac{1}{4}-\tfrac{1}{5}\right)+\cdots+\left(\tfrac{1}{999{,}999}-\tfrac{1}{1{,}000{,}000}\right)\right]$$