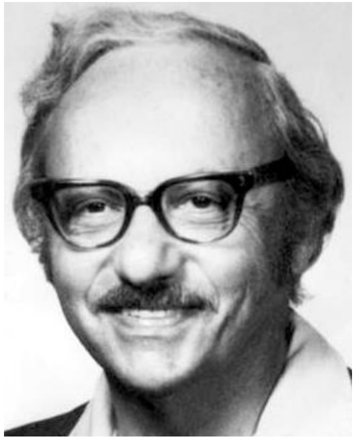
Solving Linear Programming Problems without Graphing:

When the number of variables is larger than 2 or the number of constraints is large, then the graphical method of solving a linear programming problem becomes difficult to impossible.

A few years after the end of World War II, linear programming problems with huge numbers of variables and constraints were being considered. A non-graphical method of solution, called the Simplex Method, was developed by George Dantzig to solve these large problems.



In this class, we'll look at the Simplex Method applied to special linear programming problems called standard maximization problems.

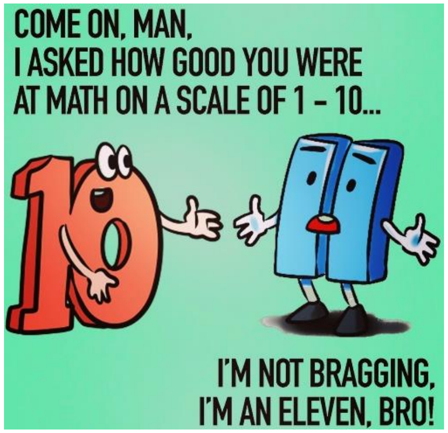
Standard Maximization Problem:

Maximize: $P = c_1x_1 + c_2x_2 + \cdots + c_nx_n$**, where** $c_1, c_2, \ldots, c_n$ **are any real numbers**

Subject to a list of constraints of the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n \le b$**, with** $b \ge 0$**, along with the nonnegative constraints** $x_1, x_2, \ldots, x_n \ge 0$**.**

The Set-up for the Simplex Method:

The original standard maximization problem is first converted into a system of linear equations, and then converted into an augmented matrix.



Example:

$$\textbf{Maximize: } P = 15x_1 + 10x_2$$

$$\textbf{Subject to } \begin{aligned} 2x_1 + x_2 &\le 10 \\ x_1 + 3x_2 &\le 10 \\ x_1, x_2 &\ge 0 \end{aligned}$$

First, the two less than or equal to inequalities are converted to equations by introducing a new variable to each equation called a slack variable: $\begin{aligned} 2x_1 + x_2 + s_1 &= 10 \\ x_1 + 3x_2 + s_2 &= 10 \end{aligned}$**.**

Next, we rearrange the objective function into the equation $-15x_1 - 10x_2 + P = 0$**, and we attach it to the previous system to get** $\begin{aligned} 2x_1 + x_2 + s_1 &= 10 \\ x_1 + 3x_2 + s_2 &= 10 \\ -15x_1 - 10x_2 + P &= 0 \end{aligned}$**, called the initial**

system. Now, we take the coefficients and constants from the initial system and put them into an augmented matrix called the initial simplex tableau:

$$\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & P & \\ \hline 2 & 1 & 1 & 0 & 0 & 10 \\ 1 & 3 & 0 & 1 & 0 & 10 \\ \hline -15 & -10 & 0 & 0 & 1 & 0 \end{array}$$

, where the column headings indicate the variables associated with the coefficients. Variables with unique unit columns(a single 1 and the rest 0's) correspond to variables, called basic variables, whose values can be read from the right-most column. For this initial simplex tableau, these variables are s_1, s_2, and *P*, whose values are $s_1 = 10, s_2 = 10, P = 0$. The other variables without unit columns, called non-basic variables, are x_1 and x_2, and in general such variables are assigned the value 0, so $x_1 = 0, x_2 = 0$. Because of the nature of a standard maximization problem, the origin will always be a corner point of the feasible region, and the Simplex Method will always begin at this corner point. The strategy of the

Simplex Method is to move from one corner point to another adjacent corner point to produce an increase in the value of the objective function. When the problem has a solution, we will eventually arrive at a corner point producing the maximum value. When the problem has no solution we will encounter difficulties along the way that will indicate the lack of a solution.

Back to the example:

$$\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & P & \\ \hline 2 & 1 & 1 & 0 & 0 & 10 \\ 1 & 3 & 0 & 1 & 0 & 10 \\ \hline -15 & -10 & 0 & 0 & 1 & 0 \end{array}$$

We are currently at the corner point $x_1 = 0, x_2 = 0$**, with an objective function value of 0, and we'd like to know if we can move to an adjacent corner point that will give the**

objective function a larger value. The way to determine if the objective function value can be increased is to look at what are called the *indicators*.

$$\begin{array}{c} \quad x_1 \qquad x_2 \quad\; s_1 \quad s_2 \quad P \qquad\quad \\ \left[\begin{array}{ccccc|c} 2 & 1 & 1 & 0 & 0 & 10 \\ 1 & 3 & 0 & 1 & 0 & 10 \\ \hline -15 & -10 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

If any of the indicators are negative, then the value of the objective function can be increased. The variable corresponding to the most negative indicator indicates the direction of movement that will increase the objective function fastest and identifies what's called the *pivot column*. For us this variable corresponding to -15 is x_1, and the interpretation is that each step in the x_1 direction will produce an increase of 15 units in the objective function. Now we need to know how far we can move in the x_1 direction until we reach an adjacent corner point. This is determined by forming ratios of the positive entries above the horizontal bar in the pivot column with the corresponding entries in the last column on the right.

$$
\begin{array}{c}
\quad x_1 \quad\; x_2 \quad s_1 \quad s_2 \quad P \\
\left[\begin{array}{ccccc|c}
2 & 1 & 1 & 0 & 0 & 10 \\
1 & 3 & 0 & 1 & 0 & 10 \\
\hline
-15 & -10 & 0 & 0 & 1 & 0
\end{array}\right]
\begin{array}{l}
\frac{10}{2} = 5 \\
\frac{10}{1} = 10 \\
\\
\end{array}
\end{array}
$$

pivot column

The smaller/smallest of these ratios determines how far we can move in the x_1 direction and still remain in the feasible region. This smallest ratio also determines the *pivot row*.

$$
\begin{array}{c}
\quad x_1 \quad\; x_2 \quad s_1 \quad s_2 \quad P \\
\left[\begin{array}{ccccc|c}
2 & 1 & 1 & 0 & 0 & 10 \\
1 & 3 & 0 & 1 & 0 & 10 \\
\hline
-15 & -10 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}
$$

pivot row

pivot column

The process of moving from one corner point to an adjacent corner point is called a *pivot*.

The intersection of the pivot column and the pivot row identifies the *pivot entry*.

$$
\begin{array}{ccccc|c}
x_1 & x_2 & s_1 & s_2 & P & \\
\left[\;\textcircled{2}\right. & 1 & 1 & 0 & 0 & 10 \\
1 & 3 & 0 & 1 & 0 & 10 \\
\hline
-15 & -10 & 0 & 0 & 1 & \left.0\;\right]
\end{array}
$$

← **pivot row**

↑ **pivot column**

The row operations that will move us from the current corner point, $(0,0)$, to the new corner point 5 units to the right, $(5,0)$, start by converting the pivot entry of 2 into a 1. Then we use this 1 to zero out the other entries in the pivot column. When we arrive at the new corner point, the objective function will have increased by $5 \cdot 15 = 75$**.**

$$\begin{array}{c} \quad x_1 \quad\; x_2 \quad\; s_1 \quad s_2 \quad P \\ \left[\begin{array}{ccccc|c} 2 & 1 & 1 & 0 & 0 & 10 \\ 1 & 3 & 0 & 1 & 0 & 10 \\ \hline -15 & -10 & 0 & 0 & 1 & 0 \end{array}\right] \\ \tfrac{1}{2}R_1 \to R_1 \end{array} \to \begin{array}{c} \left[\begin{array}{ccccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 5 \\ 1 & 3 & 0 & 1 & 0 & 10 \\ \hline -15 & -10 & 0 & 0 & 1 & 0 \end{array}\right] \\ -R_1+R_2 \to R_2, 15R_1+R_3 \to R_3 \end{array} \to \left[\begin{array}{ccccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 5 \\ 0 & \frac{5}{2} & -\frac{1}{2} & 1 & 0 & 5 \\ \hline 0 & -\frac{5}{2} & \frac{15}{2} & 0 & 1 & 75 \end{array}\right]$$

Now that the pivot is complete, you can see that the new values of the variables are $x_1 = 5, x_2 = 0, P = 75, s_1 = 0, s_2 = 5$. We have a negative indicator of $-\frac{5}{2}$, which identifies the next pivot column as involving the variable x_2. Let's find out how far we can move in the x_2 direction.

$$\begin{array}{c} x_1 \quad x_2 \quad\; s_1 \quad\; s_2 \quad P \\ \left[\begin{array}{ccccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 5 \\ 0 & \boxed{\frac{5}{2}} & -\frac{1}{2} & 1 & 0 & 5 \\ \hline 0 & -\frac{5}{2} & \frac{15}{2} & 0 & 1 & 75 \end{array}\right] \end{array} \begin{array}{l} \frac{5}{\frac{1}{2}} = 10 \\ \frac{5}{\frac{5}{2}} = 2 \\ \end{array}$$

pivot row

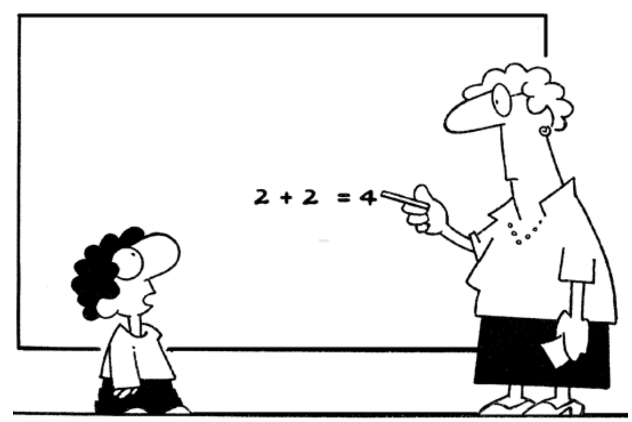
pivot column

We can move 2 units in the x_2 direction with each unit producing an increase of $\frac{5}{2}$ in the objective function, so when we get to the next corner point, the objective function value will be 80. Let's do the next pivot.

$$\begin{matrix} x_1 & x_2 & s_1 & s_2 & P \end{matrix}$$

$$\left[\begin{array}{ccccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 5 \\ 0 & \frac{5}{2} & -\frac{1}{2} & 1 & 0 & 5 \\ \hline 0 & -\frac{5}{2} & \frac{15}{2} & 0 & 1 & 75 \end{array}\right] \xrightarrow[\frac{2}{5}R_2 \to R_2]{} \left[\begin{array}{ccccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 5 \\ 0 & 1 & -\frac{1}{5} & \frac{2}{5} & 0 & 2 \\ \hline 0 & -\frac{5}{2} & \frac{15}{2} & 0 & 1 & 75 \end{array}\right] \xrightarrow[-\frac{1}{2}R_2+R_1 \to R_1,\ \frac{5}{2}R_2+R_3 \to R_3]{} \left[\begin{array}{ccccc|c} 1 & 0 & \frac{3}{5} & -\frac{1}{5} & 0 & 4 \\ 0 & 1 & -\frac{1}{5} & \frac{2}{5} & 0 & 2 \\ \hline 0 & 0 & 7 & 1 & 1 & 80 \end{array}\right]$$

None of the indicators are negative, so we've reached the maximum value of the objective function. The maximum value of *P* is 80, and it occurs when $x_1 = 4, x_2 = 2$.



"You're certainly entitled to your opinion."

When a student tells you that they didn't know they needed to show their work.

More examples:

1. Maximize: $P = 5x_1 + 2x_2 - x_3$

Subject to

$$\begin{aligned} x_1 + x_2 - x_3 &\le 10 \\ 2x_1 + 4x_2 + 3x_3 &\le 30 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & P & \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 1 & -1 & 1 & 0 & 0 & 10 \\ 2 & 4 & 3 & 0 & 1 & 0 & 30 \\ \hline -5 & -2 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

2. Maximize: $P = 2x_1 + 3x_2$

Subject to $-2x_1 + x_2 \le 4$

$$x_2 \le 10$$

$$x_1, x_2 \ge 0$$

$$\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & P & \\ \end{array}$$

$$\left[\begin{array}{ccccc|c} -2 & 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 0 & 10 \\ \hline -2 & -3 & 0 & 0 & 1 & 0 \end{array}\right]$$