Measuring Matrices:

The size of a matrix is expressed in a similar way as the size of a board of lumber. The size of a matrix is called its dimensions, and it's the $(\#\text{ of rows})\times(\#\text{ of columns})$**.**

Examples: Find the dimensions of the following matrices.

1. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 2 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$

If the number of rows and the number of columns are the same, then the matrix is called a *square matrix*. A matrix with any dimensions whose entries are all zeros is called a *zero matrix*.

Operations with Matrices:

Matrices can be viewed in a similar way as numbers, and as such, arithmetic operations can be performed with matrices.

Addition and Subtraction of Matrices:

If two matrices have the same dimensions, then they can be added or subtracted by adding or subtracting corresponding entries in the two matrices. The result of addition or subtraction of two matrices will be a matrix with the same dimensions.

Examples: Perform the following operations, if possible.

1. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 2 & -3 \\ 4 & -5 \end{bmatrix}$

$$\begin{bmatrix} G & A & L & L & E \\ R & Y & S & A & V \\ E & S & Y & O & U \\ M & O & N & E & Y \end{bmatrix}$$

Matrix Mack

2. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 2 & 0 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 0 & 2 \\ -1 & 0 & -2 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 1 \\ 2 & -3 & 1 \end{bmatrix}$

Multiplication of a Number and a Matrix:

Any number can be multiplied with any matrix by simply multiplying each entry of the matrix with the number. The result will always be a matrix of the same dimensions of the original matrix.

Examples:

1. $2\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

2. $-3\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 2 & 0 \end{bmatrix}$

3. $3\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} - 2\begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$

A basic application of matrix operations:

There are three convenience stores in a small town. This week, Store I sold 88 loaves of bread, 48 quarts of milk, 16 jars of peanut butter, and 112 pounds of cold cuts. Store II sold 105 loaves of bread, 72 quarts of milk, 21 jars of peanut butter, and 147 pounds of cold cuts. Store III sold 60 loaves of bread, 40 quarts of milk, no peanut butter, and 50 pounds of cold cuts.

Here's the 3×4 **matrix that expresses the sales information for the three stores.**

$$\begin{matrix} & B & M & P & C & \\ A = & \begin{bmatrix} 88 & 48 & 16 & 112 \\ 105 & 72 & 21 & 147 \\ 60 & 40 & 0 & 50 \end{bmatrix} & & & & \begin{matrix} Store\ I \\ Store\ II \\ Store\ III \end{matrix} \end{matrix}$$

During the following week, sales on these products at Store I increased by 25%, sales at Store II increased by $\frac{1}{3}$, and sales at Store III increased by 10%.

Here's the sales matrix for that week.

$$
\begin{array}{c} \\ D = \end{array}
\begin{array}{c}
\begin{array}{cccc} B & M & P & C \end{array} \\
\begin{bmatrix} 110 & 60 & 20 & 140 \\ 140 & 96 & 28 & 196 \\ 66 & 44 & 0 & 55 \end{bmatrix}
\end{array}
\begin{array}{l} \\ \textit{Store I} \\ \textit{Store II} \\ \textit{Store III} \end{array}
$$

What matrix operation involving *A* and *D* would produce a matrix that represents the total sales for the three stores for these two weeks?

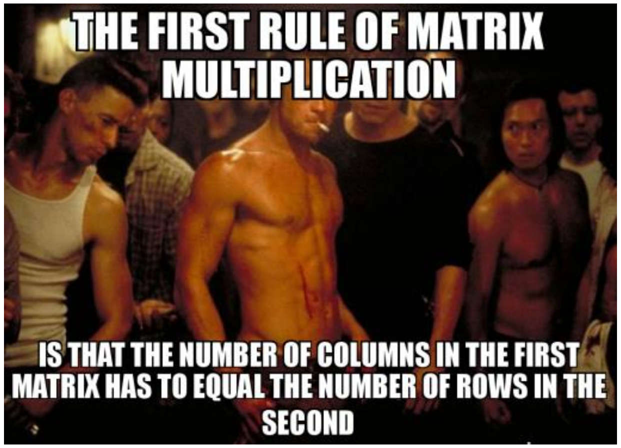
Multiplication of two Matrices:

If matrix *A* has dimensions $n \times k$ and matrix *B* has dimensions $k \times m$, then the matrix product AB is defined and has dimensions $n \times m$. The $(i, j)^{\text{th}}$ entry of AB is found by multiplying the entries of the *i*th row of *A* with the entries of the *j*th column of *B* and adding the products.

For example: If you wanted to perform the product $\underset{2\times 2}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}\underset{2\times 2}{\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}}$**, it would make sense because the number of columns of the first matrix is equal to the number of rows of the second matrix, and the product matrix would have dimensions of 2×2. To get the entry in the product matrix in the first row and first column, we'd multiply and add the first row of** $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **with the first column of** $\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$ **to get** $1 \cdot 1 + 2 \cdot 2 = 5$**.**

So we know this much of the product matrix so far: $\begin{bmatrix} 5 & \\ & \end{bmatrix}$**. To get the entry in the**

product matrix in the first row and second column we'd multiply and add the first row of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **with the second column of** $\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$ **to get** $1 \cdot (-1) + 2 \cdot (-2) = -5$**. So we know this much of the product matrix so far:** $\begin{bmatrix} 5 & -5 \\ & \end{bmatrix}$**. If we continue this process, we'll end up with a product matrix of** $\begin{bmatrix} 5 & -5 \\ 11 & -11 \end{bmatrix}$**. One of the big differences between matrix multiplication and multiplication with numbers is that for matrices, order is important. Sometimes the matrix product *AB* makes sense, but *BA* doesn't make sense, and even if both orders are defined, they might not produce matrices that are equal. In our first example, the reverse order product makes sense** $\underset{2\times 2}{\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} = \underset{2\times 2}{\begin{bmatrix} -2 & -2 \\ -4 & -4 \end{bmatrix}}$**, but it isn't equal to the original order product. So for matrices, unlike numbers, order is important in multiplication.**



More Examples:

1. $$\underset{3\times 2}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}}\underset{2\times 3}{\begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 1 \end{bmatrix}}$$

2. $$\underset{3\times 2}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}}\underset{2\times 2}{\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}}$$

$$\begin{pmatrix} 3 & 4 \\ 8 & 7 \end{pmatrix}\begin{pmatrix} 7 & 2 \\ 4 & 9 \end{pmatrix} = \begin{pmatrix} 37 & 42 \\ 84 & 79 \end{pmatrix}$$

3. $$\underset{2\times 3}{\begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 1 \end{bmatrix}}\underset{3\times 2}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}}$$

4. $$\underset{2\times 2}{\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}}\underset{3\times 2}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}}$$

Another basic application of matrix operations:

Burger Barn's three locations sell hamburgers, fries, and soft drinks. Barn I sells 900 burgers, 600 orders of fries, and 750 soft drinks each day. Barn II sells 1500 burgers, 950 orders of fries, and 900 soft drinks each day. Barn III sells 1150 burgers, 800 orders of fries, and 825 soft drinks each day.

Here's the 3×3 matrix, *S*, that represents the sales figures for all three locations.

$$\begin{array}{cccc} & \textit{Barn I} & \textit{Barn II} & \textit{Barn III} & \\ S = & \left[\begin{array}{ccc} 900 & 1500 & 1150 \\ 600 & 950 & 800 \\ 750 & 900 & 825 \end{array}\right] & & & \begin{array}{l} \textit{Burger} \\ \textit{Fries} \\ \textit{Soft Drink} \end{array} \end{array}$$

Burgers cost \$2.00 each, fries \$1.50 an order, and soft drinks \$1.00 each.

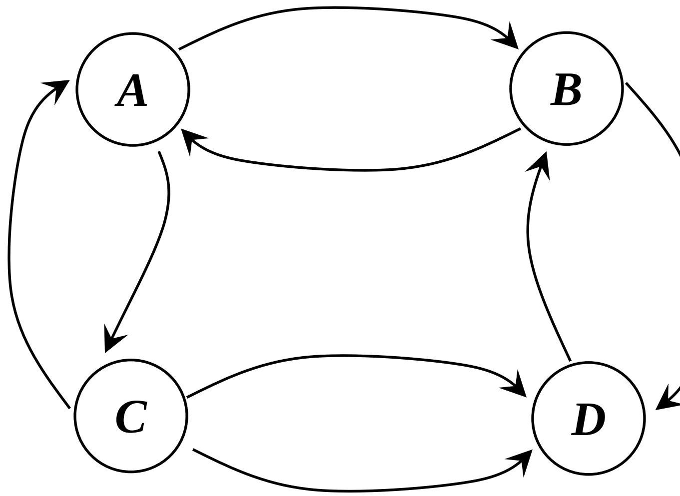
Here's the 1×3 matrix *P* that displays the prices.

$$P = \begin{bmatrix} \overset{\textit{Burger}}{2} & \overset{\textit{Fries}}{1.5} & \overset{\textit{Drink}}{1} \end{bmatrix}$$

What matrix product involving *S* and *P* displays the daily revenue at each of the three locations?*(Be careful about the dimensions!)*

Here's a more advanced application of matrix multiplication:

The following diagram illustrates the flights among four cities.



The flights are recorded in the matrix, *T*, called an incidence or connection matrix.

$$\begin{array}{cc} & \text{Destination} \\ & \begin{array}{cccc} A & B & C & D \end{array} \\ \text{Origin} \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix} = T \end{array}$$

Each entry indicates the number of direct flights from the city of origin to the city of destination.

Although you could figure out the number of 1-stop flights among the cities, it can be calculated by simply squaring the matrix *T*.

$$T^2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 3 \\ 0 & 2 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

For example, the 3 in the third row and second column indicates that there are three 1-stop flights from city *C* to city *B*.

Cubing matrix, *T*, will determine the numbers of 2-stop flights.

$$T^3 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 3 \\ 0 & 2 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 5 & 2 & 0 \\ 3 & 0 & 0 & 4 \\ 4 & 0 & 0 & 5 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

If we keep a running total of the powers of *T*, then we can determine the fewest number of flights needed to go from one city to another:

$$T = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$, so you can't get from *A* to *D*, *B* to *C*, *C* to *B*, *D* to *A*, or *D* to *C* in just 1 flight.

$$T + T^2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 & 3 \\ 0 & 2 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & 3 \\ 1 & 2 & 1 & 1 \\ 1 & 3 & 1 & 2 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$, so you can't get from *D* to *C* in in 2 or fewer flights.

$$T + T^2 + T^3 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 & 3 \\ 0 & 2 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 5 & 2 & 0 \\ 3 & 0 & 0 & 4 \\ 4 & 0 & 0 & 5 \\ 0 & 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 3 & 3 \\ 4 & 2 & 1 & 4 \\ 5 & 3 & 1 & 7 \\ 1 & 3 & 1 & 1 \end{bmatrix}$$, so you can get from each city to another city in 3 or fewer flights.