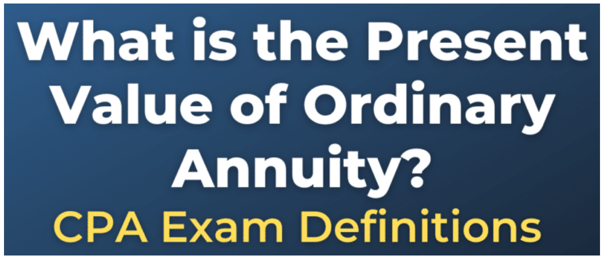
Present Value of an Annuity:

The present value of an annuity is the amount of money that you could deposit right now into the account that would produce the same amount of money as the annuity does after *n* payments.



$$\underbrace{PV(1+i)^n}_{\text{money produced from the present value}} = \underbrace{Pmt\left[\frac{(1+i)^n - 1}{i}\right]}_{\text{money produced from the } n \text{ annuity payments}}$$

So

$$PV = Pmt\left[\frac{1-\dfrac{1}{(1+i)^n}}{i}\right]$$

Or

$$PV = Pmt\left[\frac{1-(1+i)^{-n}}{i}\right]$$

Example:

Find the present value of an annuity with monthly payments of $500 into an account paying 3.1% compounded monthly for 6 years.

$$PV = 500\left[\frac{1-\left(1+\frac{.031}{12}\right)^{-72}}{\frac{.031}{12}}\right]$$

Another interpretation of the present value is that it's the amount of money you can deposit now into the account that will allow you to make *n* equal periodic withdrawals from the account so that the n^{th} withdrawal empties the account.

Example:

How much money should you deposit now into an account paying 2.7% compounded semi-annually that will allow you to make semi-annual withdrawals of $2,000 for the next 10 years with your last withdrawal emptying the account?

$$PV = 2000\left[\frac{1-\left(1+\frac{.027}{2}\right)^{-20}}{\frac{.027}{2}}\right]$$

Loan Repayment(Amortization):

When you borrow money, you are acting like a bank that is holding a deposit that is earning compound interest. The company that loaned you the money will make *n* equal periodic withdrawals(your payments) until the money they deposited with you is zeroed out by the last withdrawal(your last payment).

We'll just solve the present value formula for *Pmt* to find a formula for your payment amount in repaying the loan.

$$PV = Pmt\left[\frac{1-(1+i)^{-n}}{i}\right]$$

$$Pmt = PV\left[\frac{i}{1-(1+i)^{-n}}\right]$$

Examples:

1. You borrow $20,000 at 3.9% compounded monthly for 5 years to buy a car. What's your monthly payment? How much extra in interest will you pay on this loan?

2. You borrow $150,000 at 3.1% compounded monthly for 30 years to buy a house. What's your monthly payment? How much extra in interest will you pay on this loan?

Amortization Tables:

Sometimes in the course of paying back a loan, not all the payments are the same. Since the exact amount of the payment must be rounded to the nearest penny, the small difference is accounted for in the final payment, which might be slightly different than all the other payments. Also, for tax purposes, it's nice to have a complete break-down of the portions of each payment that go toward reducing the principal(amount borrowed) and paying off the earned interest. All of these goals can be achieved with an amortization table.

Examples: 1. Create an amortization table for a loan of $1,000 at 3% to be repaid with 6 monthly payments.

The first thing we'll do is calculate the monthly payment: $1{,}000\left[\frac{\frac{.03}{12}}{1-\left(1+\frac{.03}{12}\right)^{-6}}\right]=$

Next, we'll create a blank amortization table that we'll fill-in as we progress through the payments.

Payment #	Payment	Principal portion	Interest portion	Remaining balance
0				
1				
2				
3				
4				
5				
6				

The row corresponding to the 0^{th} payment is all blank except for the remaining balance, which is the amount of the loan. Also, all the payments will be the same except possibly the 6^{th} payment, and the remaining balance after the 6^{th} payment will be $0.

Payment #	Payment	Principal portion	Interest portion	Remaining balance
0	-	-	-	$1,000
1	$168.13			
2	$168.13			
3	$168.13			
4	$168.13			
5	$168.13			
6				$0

The next thing we'll do is calculate the interest portion of the first payment by multiplying the remaining balance of $1,000 by the interest rate per month, .03/12. The principal portion is the payment amount minus the interest portion. The new remaining balance is the original balance of $1,000 minus the principal portion of the payment.

Payment #	Payment	Principal portion	Interest portion	Remaining balance
0	-	-	-	$1,000
1	$168.13	$165.63	$2.50	$834.37
2	$168.13			
3	$168.13			
4	$168.13			
5	$168.13			
6				$0

This process is continued through the next-to-last payment, the 5th payment.

Payment #	Payment	Principal portion	Interest portion	Remaining balance
0	-	-	-	$1,000
1	$168.13	$165.63	$2.50	$834.37
2	$168.13	$166.04	$2.09	$668.33
3	$168.13	$166.46	$1.67	$501.87
4	$168.13	$166.88	$1.25	$334.99
5	$168.13	$167.29	$.84	$167.70
6				$0

Now for the 6th payment, we know that the principal portion must equal the remaining balance of $167.70, so to get the 6th payment we'll add the interest portion to the $167.70.

Payment #	Payment	Principal portion	Interest portion	Remaining balance
0	-	-	-	$1,000
1	$168.13	$165.63	$2.50	$834.37
2	$168.13	$166.04	$2.09	$668.33
3	$168.13	$166.46	$1.67	$501.87
4	$168.13	$166.88	$1.25	$334.99
5	$168.13	$167.29	$.84	$167.70
6	$168.12	$167.70	$.42	$0

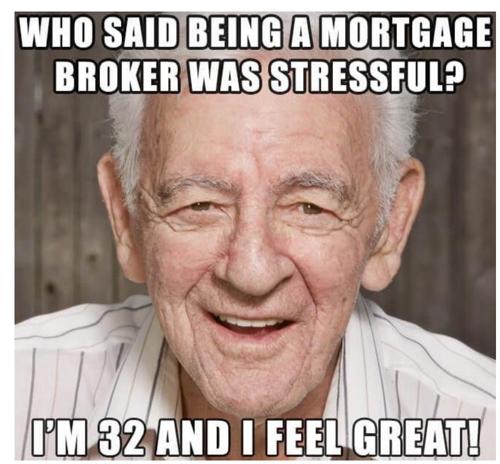
So you can see that the final payment is slightly different than the others. Also, if we calculated the total interest paid by multiplying the payment of $168.13 by 6 and subtracting the loan amount of $1,000, we'd be off by a penny from the actual interest amount of $8.77.

2. Create an amortization table for a loan of $5,000 at 4% to be repaid with 5 monthly payments.

The first thing we'll do is calculate the monthly payment: $5{,}000\left[\dfrac{\frac{.04}{12}}{1-\left(1+\frac{.04}{12}\right)^{-5}}\right]=$

Payment #	Payment	Principal portion	Interest portion	Remaining balance
0	-	-	-	$5,000
1				
2				
3				
4				
5				$0

In general, the principal portions of the payments will increase, while the interest portions will decrease.



Scientific Calculator Advice:

1. How much money should you deposit now into an account paying 2.7% compounded semi-annually that will allow you to make semi-annual withdrawals of $2,000 for the next 10 years with your last withdrawal emptying the account?

The formula that we will use to find the amount of money is $PV = Pmt\left[\frac{1-\left(1+\frac{r}{m}\right)^{-n}}{\frac{r}{m}}\right]$. We need to plug in the correct values and get the calculator to cooperate. Let's start with plugging in the correct values: The payment amount is $2,000. The annual interest rate as a decimal is .027. The number of payments/compounding-periods in one year is 2. The total number of payments/compounding-periods is 20. Therefore, $PV = 2{,}000\left[\frac{1-\left(1+\frac{.027}{2}\right)^{-20}}{\frac{.027}{2}}\right]$. Now for the calculator. Using a standard scientific calculator, I'd start by typing in .027, pressing $\div$, typing 2 and pressing =. Then I would press the + key, type 1 and press =. We have to raise this to the negative 20^{th} power, so I'd press either ^ or the x^y key(depending on the calculator), type in 20, press the change sign key and press =. Next, I'd press the change sign key, press the + key, type 1 and press =. Then I'd press $\div$, type .027 and press =. Then I'd press $\times$, type 2 and press =. Finally, I'd press $\times$, type 2000 and press =. You should be looking at 34850.4191 on your calculator display. This needs to be converted into dollar and cent format rounded to the nearest penny, which would be $34,850.42.

2. You borrow \$20,000 at 3.9% compounded monthly for 5 years to buy a car. What's your monthly payment?

The formula that we will use to find the payment amount is $Pmt = PV\left[\frac{\frac{r}{m}}{1-\left(1+\frac{r}{m}\right)^{-n}}\right]$. We need to plug in the correct values and get the calculator to cooperate. Let's start with plugging in the correct values: The present value amount is \$20,000. The annual interest rate as a decimal is .039. The number of payments/compounding-periods in one year is 12. The total number of payments/compounding-periods is 60. Therefore, $Pmt = 20,000\left[\frac{\frac{.039}{12}}{1-\left(1+\frac{.039}{12}\right)^{-60}}\right]$. Now for the calculator. Using a standard scientific calculator, I'd start by typing in .039, pressing ÷, typing 12 and pressing =. Then I would press +, type 1 and press =. We have to raise this to the negative 60^{th} power, so I'd press either ^ or the x^y key(depending on the calculator), type in 60, press the change sign key and press =. Next, I'd press the change sign key, press the + key, type 1 and press =. So far we've calculated the value of the denominator, but the calculator doesn't know that this value is the denominator. To move this value into the denominator, we need to press the either the $\frac{1}{x}$ or the x^{-1} key(depending on the calculator). Then I'd press ×, type .039 and press =. Then I'd press ÷, type 12 and press =. Finally, I'd press × , type 20,000 and press =. You should be looking at 367.4285788 on your calculator display. This needs to be converted into dollar and cent format rounded to the nearest penny, which would be \$367.43.