Review of Reduced Rational Functions:

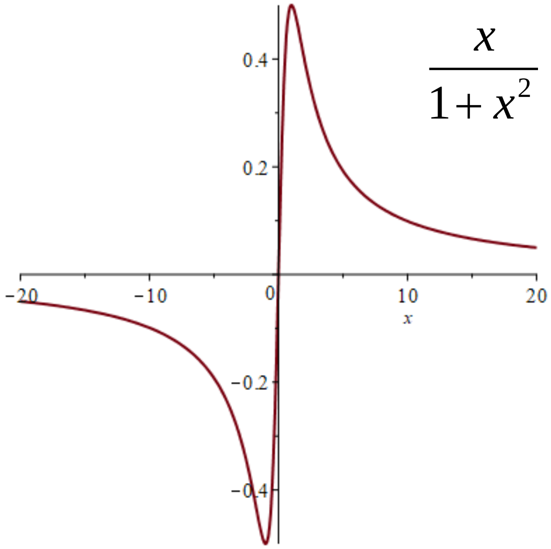
$f(x)=\dfrac{p(x)}{q(x)}$, **where** $p(x)$ **and** $q(x)$ **have no common real zeros.** ***If they have common imaginary zeros, then the corresponding common factors should be cancelled out.***

Asymptotes:

Lines that the graph of a rational function approach.

Horizontal:

If the degree of $p(x)$ **is less than the degree of** $q(x)$**, then** $y=0$**(*x*-axis) is the horizontal asymptote.**



$$\frac{x}{1+x^2}$$

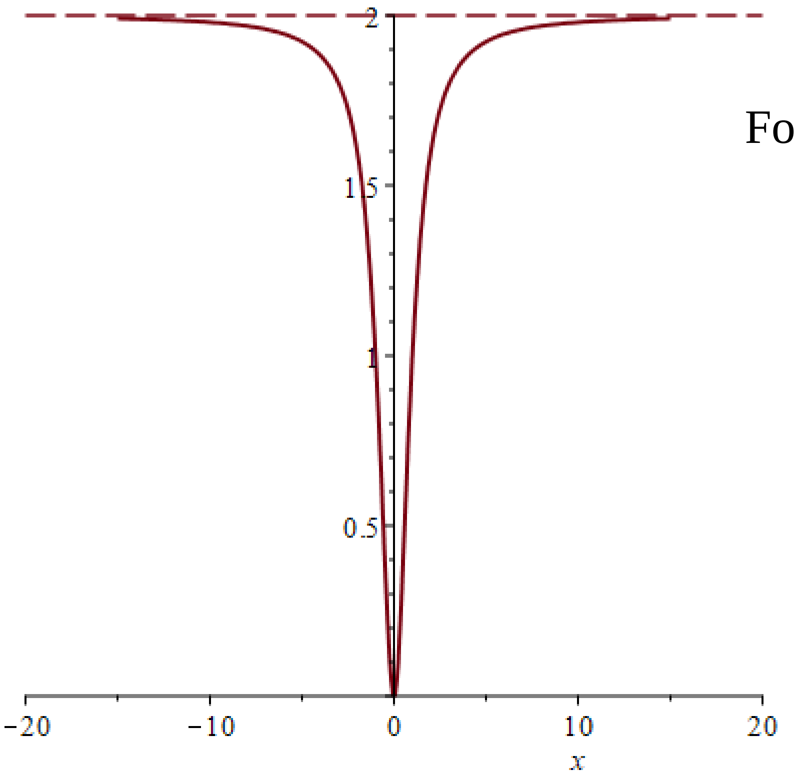
$$\text{For } |x| \text{ large, } \frac{x}{1+x^2}=\frac{\frac{x}{x^2}}{\frac{1}{x^2}+\frac{x^2}{x^2}}=\frac{\overset{\nearrow 0}{\frac{1}{x}}}{\underset{\searrow 0}{\frac{1}{x^2}}+1}\approx\frac{0}{0+1}=0$$

Divide everything by the highest power of x in the denominator.

If the degree of $p(x)$ is equal to the degree of $q(x)$, then

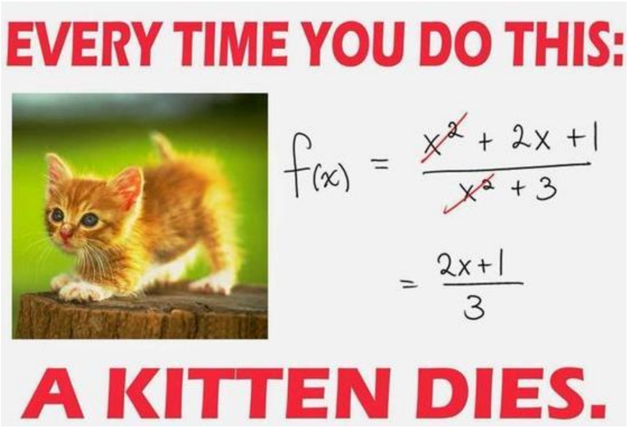
$$y=\frac{\text{leading coefficient of } p(x)}{\text{leading coefficient of } q(x)}$$ **is the horizontal asymptote.**

$$\frac{2x^2}{1+x^2}$$



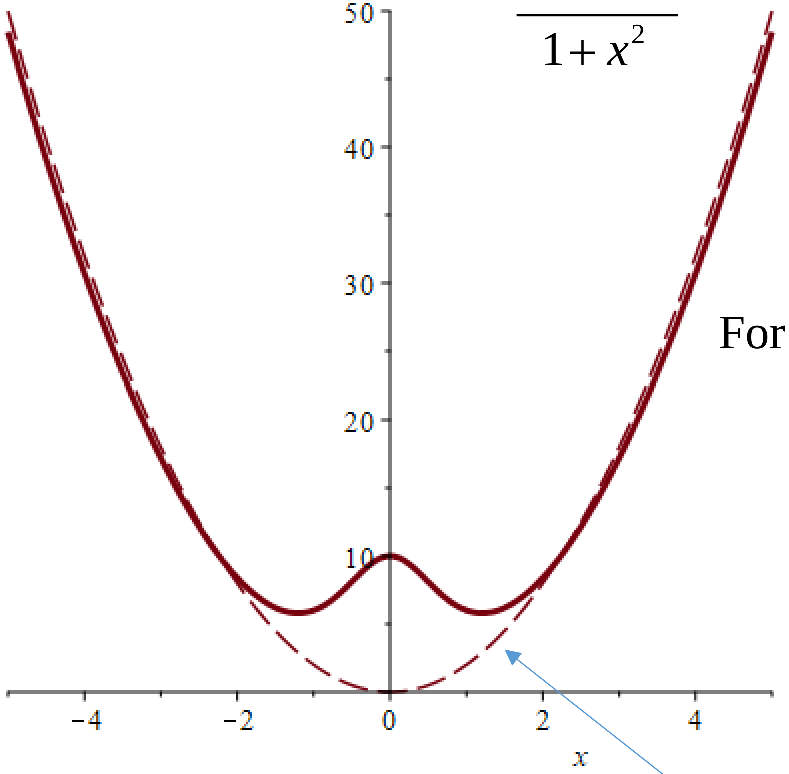
$$\text{For } |x| \text{ large,} \frac{2x^2}{1+x^2}=\frac{\frac{2x^2}{x^2}}{\frac{1}{x^2}+\frac{x^2}{x^2}}=\frac{2}{\underbrace{\frac{1}{x^2}}_{0}+1}\approx\frac{2}{0+1}=2$$

Divide everything by the highest power of x in the denominator.



If the degree of $p(x)$ is greater than the degree of $q(x)$, then there is no horizontal asymptote. The end behavior is same as the end behavior of the polynomial $\frac{\text{highest degreed term in } p(x)}{\text{highest degreed term in } q(x)}$.

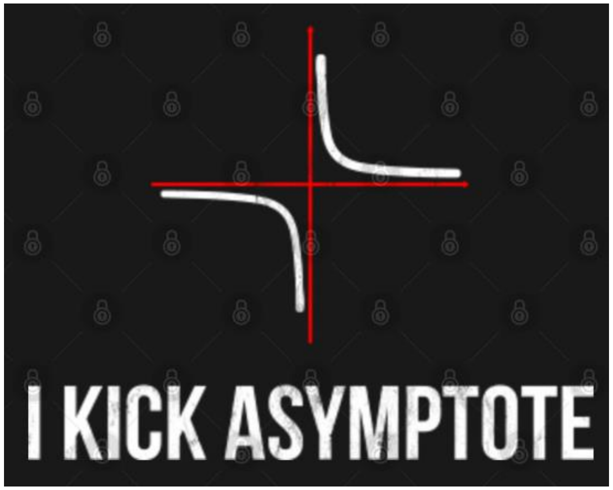
$$\frac{2x^4+10}{1+x^2}$$



For $|x|$ large, $\frac{2x^4+10}{1+x^2}=\frac{\frac{2x^4}{x^2}+\frac{10}{x^2}}{\frac{1}{x^2}+\frac{x^2}{x^2}}=\frac{2x^2+\overset{0}{\boxed{\frac{10}{x^2}}}}{\underset{0}{\boxed{\frac{1}{x^2}}}+1}\approx\frac{2x^2+0}{0+1}=2x^2$

Divide everything by the highest power of x in the denominator.

End behavior is the same as $\frac{2x^4}{x^2}=2x^2$**.**



Examples:

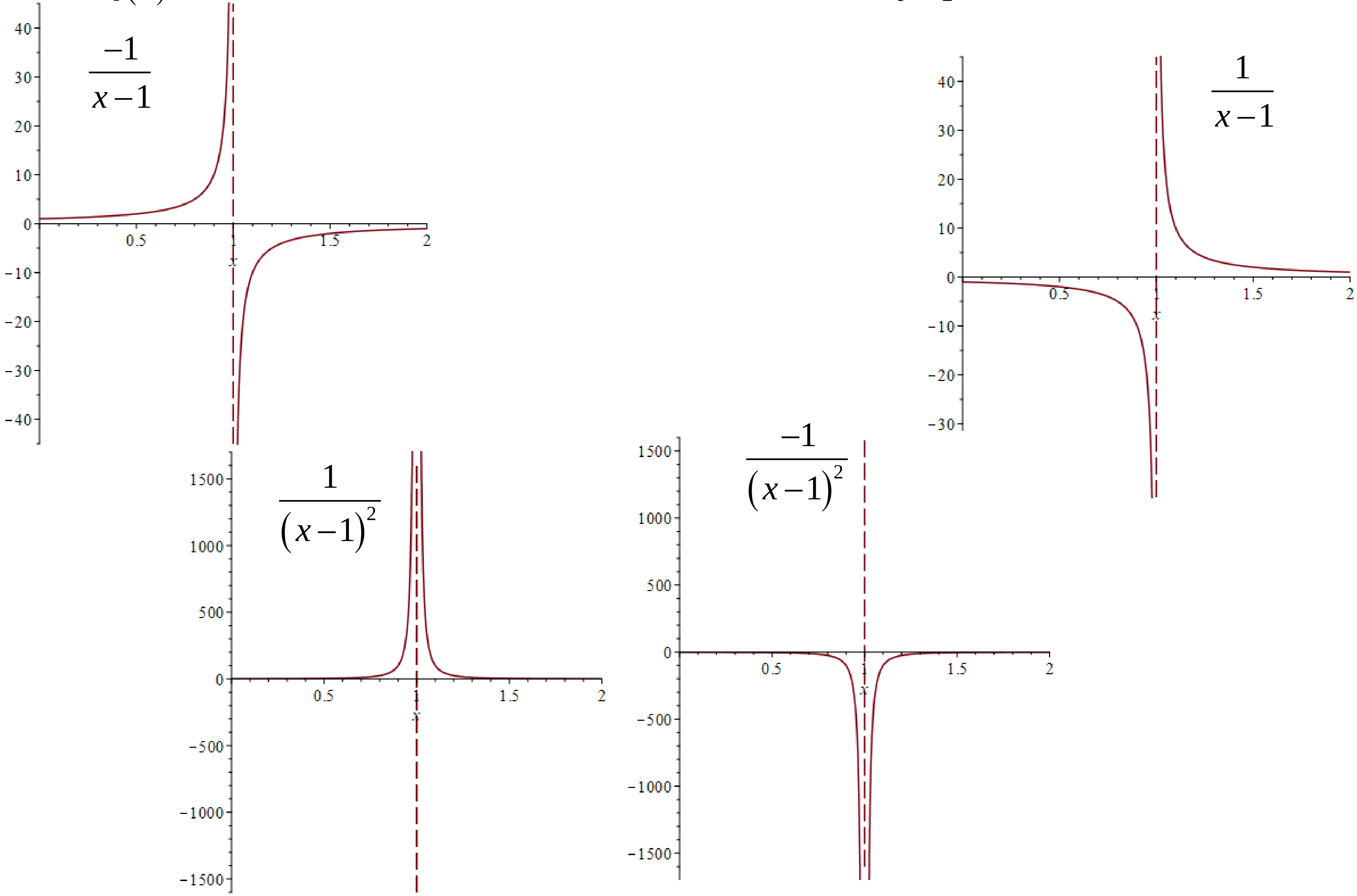
1. $f\left(x\right)=\dfrac{2x^3+x-1}{3x^3+x^2}$ **H.A.:**

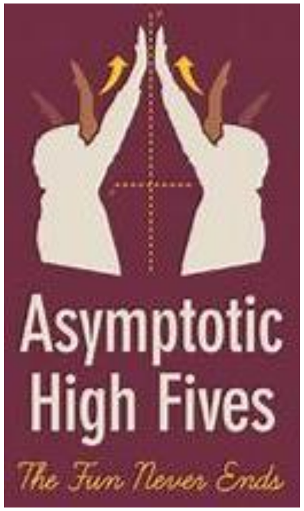
2. $f\left(x\right)=\dfrac{2x^2+x-1}{3x^3+x^2}$ **H.A.:**

3. $f\left(x\right)=\dfrac{2x^6+x-1}{3x^3+x^2}$ **H.A.:**

Vertical:

If $q(c)=0$, for *c* a real number, then $x=c$ is a vertical asymptote.



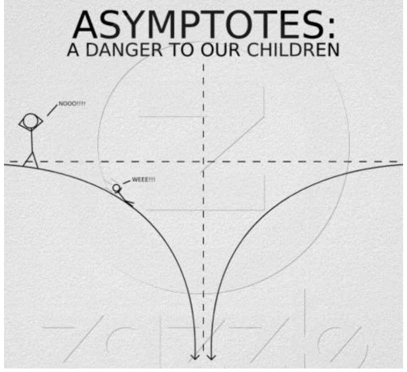


Examples:

1. $f\left(x\right)=\dfrac{3x+5}{x-6}$ **V.A.:**

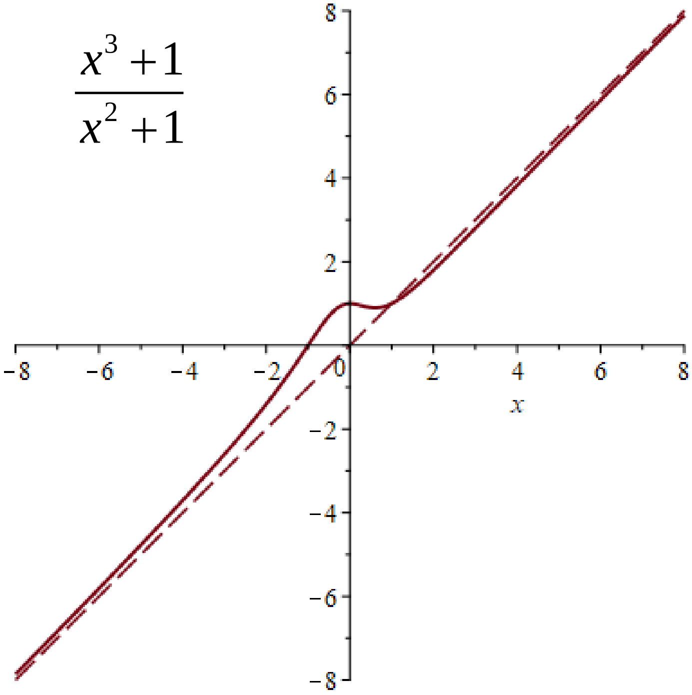
2. $f\left(x\right)=\dfrac{x^{3}+1}{x^{2}-5x-14}$ **V.A.:**

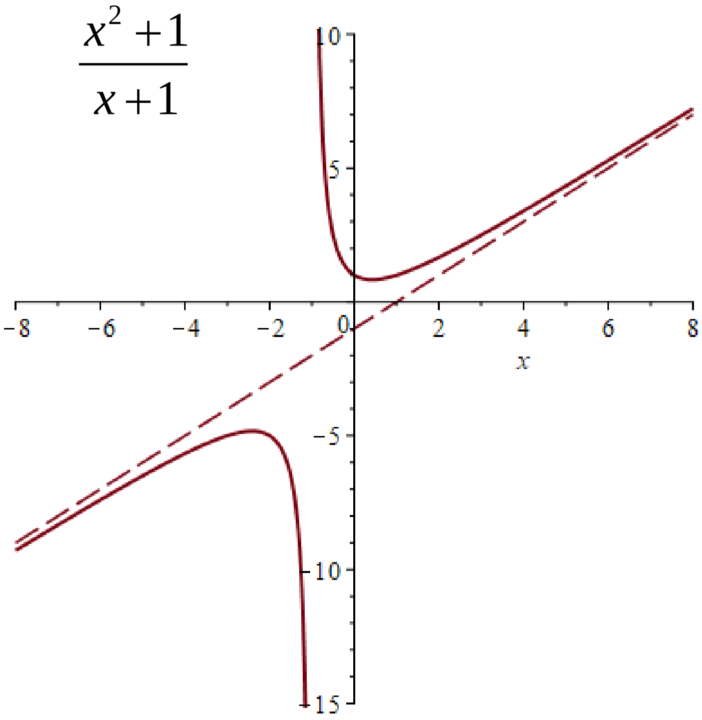
3. $f\left(x\right)=\dfrac{x^{3}}{x^{4}-1}$ **V.A.:**

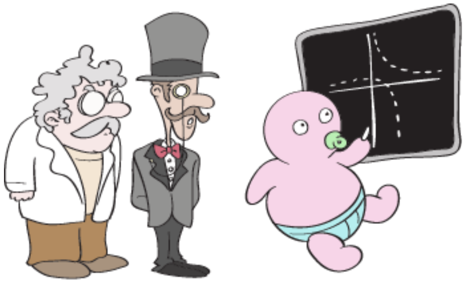


<u>Slant/Oblique:</u>

If the degree of $p(x)$ is 1 more than the degree of $q(x)$, then the rational function will have a slant/oblique asymptote. The equation of the slant asymptote can be determined using polynomial long division.



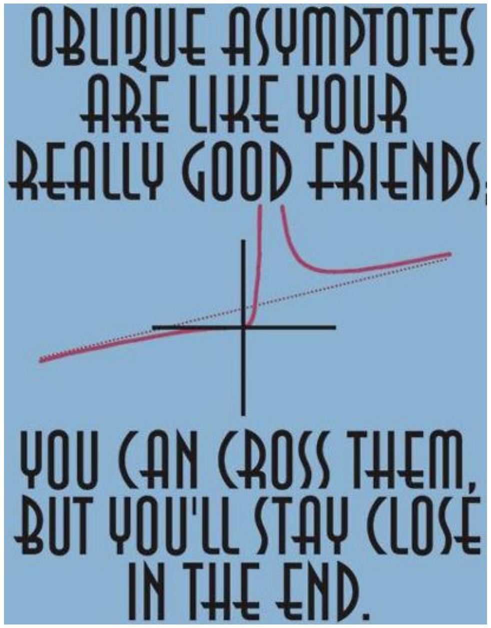




Examples:

1. $f\left(x\right)=\dfrac{2x^2+x-1}{x+2}$ $\qquad x+2\overline{\big)2x^2+x-1}$ $\qquad$ **S.A.:**

2. $f\left(x\right)=\dfrac{x^3+8}{x^2-5x+6}$ $\qquad x^2-5x+6\overline{\big)x^3+8}$ $\qquad$ **S.A.:**



Sign Charts for Rational Functions:

The behavior of a rational function in the vicinity of its vertical asymptotes can be determined by the sign of the function values. To make a sign chart for a rational function, draw a number line and locate the zeros of the numerator and label them with a *0*, since the function value is zero there. Locate the zeros of the denominator, and label them with a *u*, since the function is undefined at these values. Use what you know about the graphs of polynomial functions to determine the sign of the rational function on the intervals in between and on the edges.

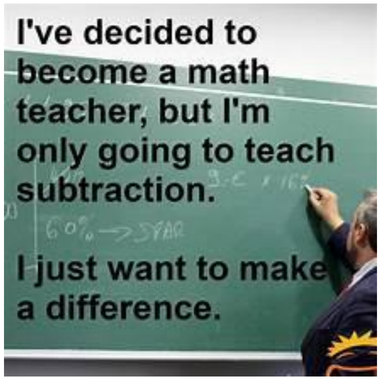
Examples:

1. $f\left(x\right)=\dfrac{3x}{x+4}$

2. $f\left(x\right)=\dfrac{6}{x^2+x-6}$

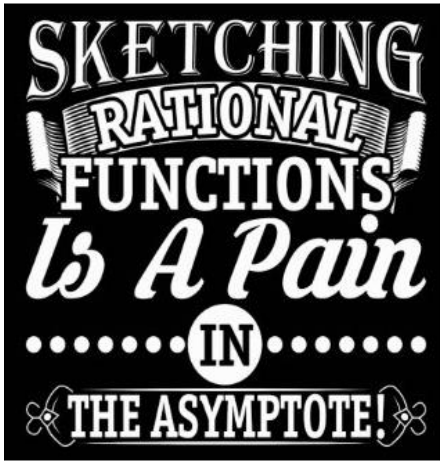
3. $f(x)=\dfrac{3x}{(x-1)(x-3)^2}$

4. $f(x)=\dfrac{2-x}{(x+1)(x-4)}$



Sketching graphs of rational functions:

1. Find and draw the horizontal, vertical, and slant asymptotes as dashed lines.



2. Find and label the *x*-intercepts as points.

3. Find and label the *y*-intercept as a point.

4. Create the sign chart.

5. Connect all the dots in a reasonable manner using the sign chart as a guide.

"Of course you have problems!
You're a math teacher."

Examples:

1. $f(x)=\dfrac{x}{(x-1)(x+2)}$

Asymptotes:

H.A.:

V.A.:

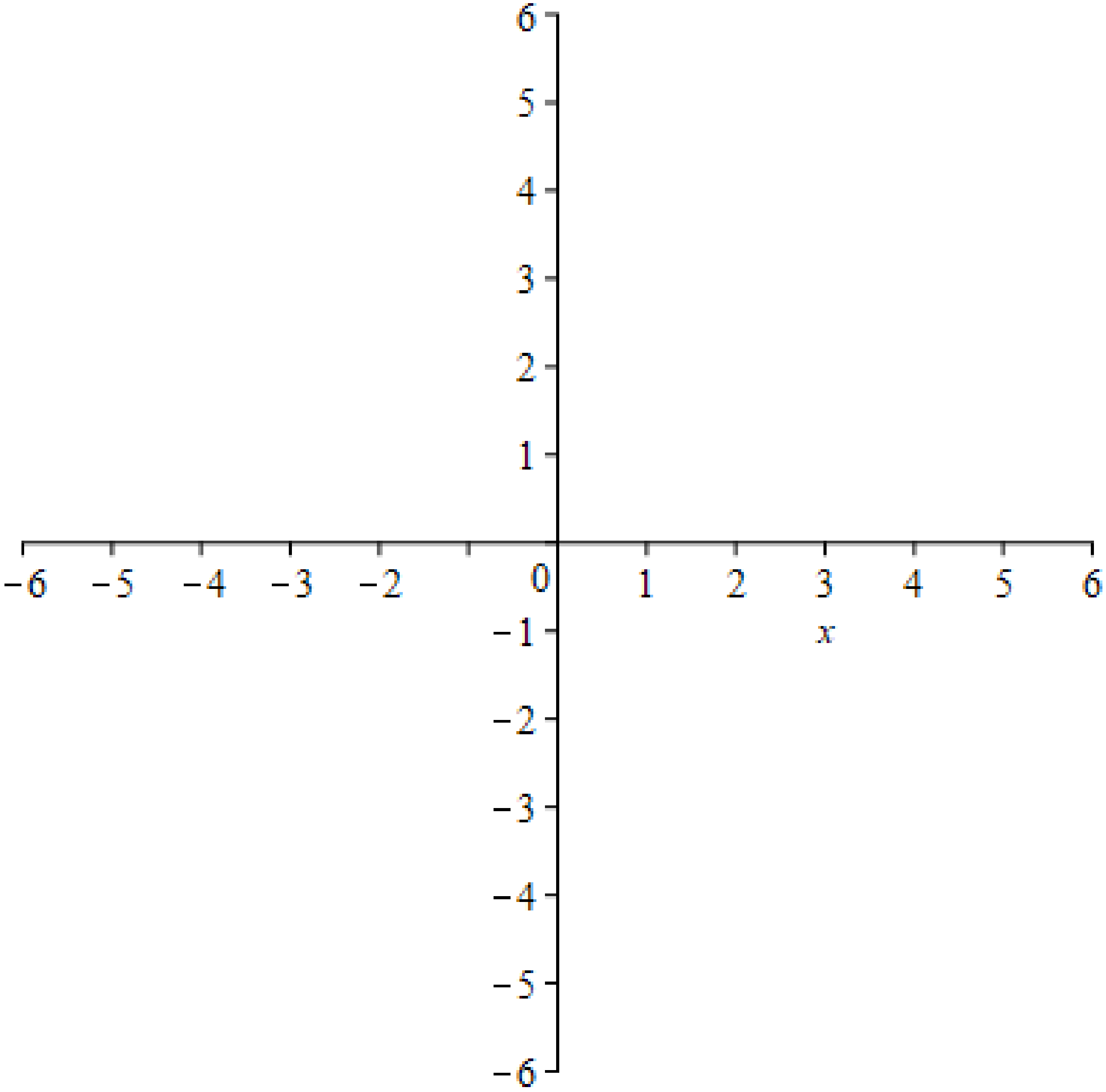
S.A.:

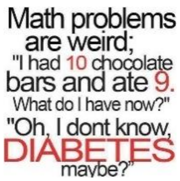
Intercepts:

***x*-int(s):**

***y*-int:**

Sign Chart:





2. $f\left(x\right)=\dfrac{2x+4}{x-1}$

Asymptotes:

H.A.:

V.A.:

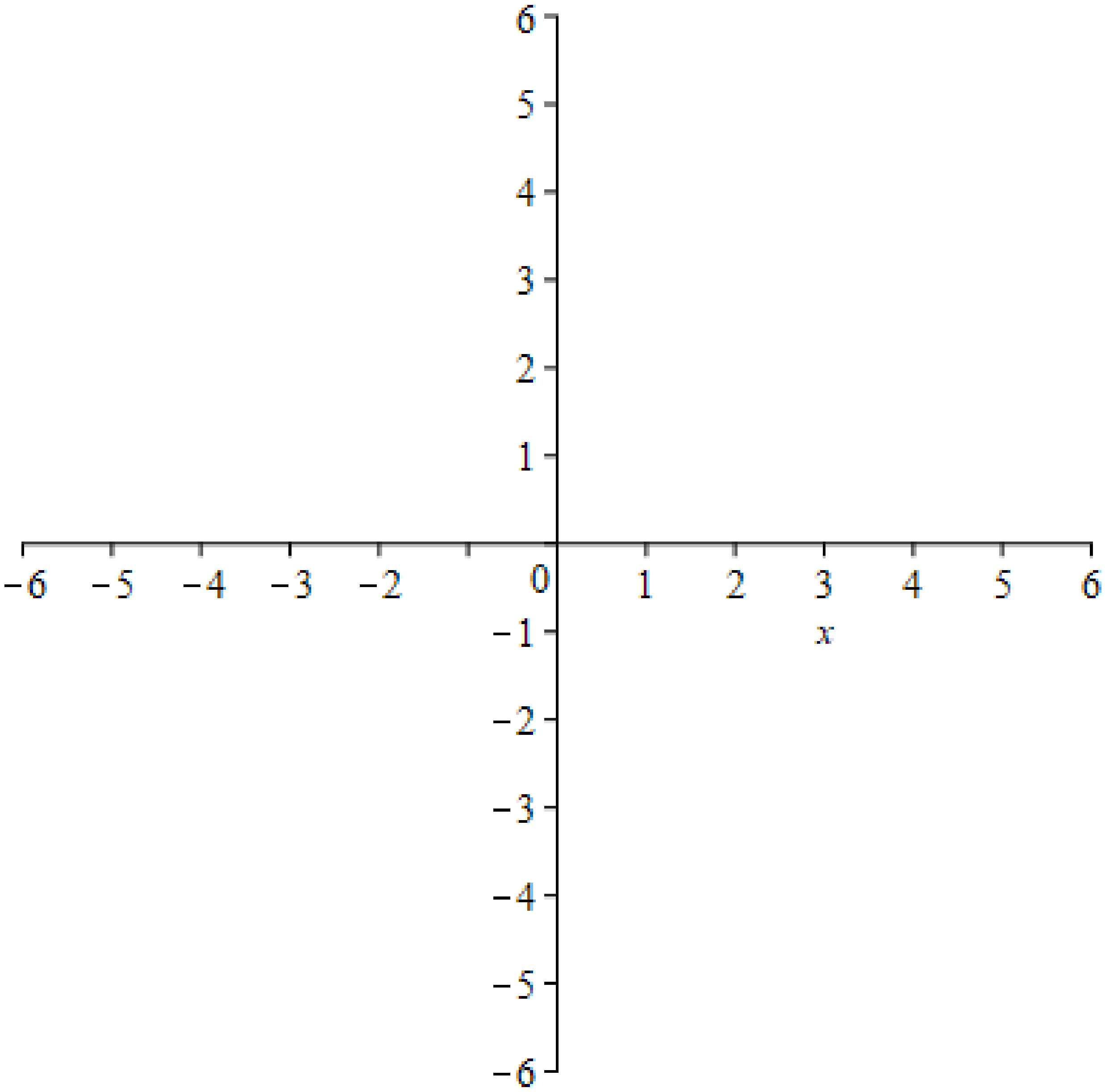
S.A.:

Intercepts:

***x*-int(s):**

***y*-int:**

Sign Chart:



NEVER DISCUSS INFINITY WITH A MATHEMATICIAN. YOU'LL NEVER HEAR THE END OF IT

3. $f\left(x\right)=\dfrac{x\left(x-1\right)^{2}\left(x^{2}+1\right)}{\left(x+3\right)^{3}\left(x^{2}+1\right)}$

Asymptotes:

H.A.:

V.A.:

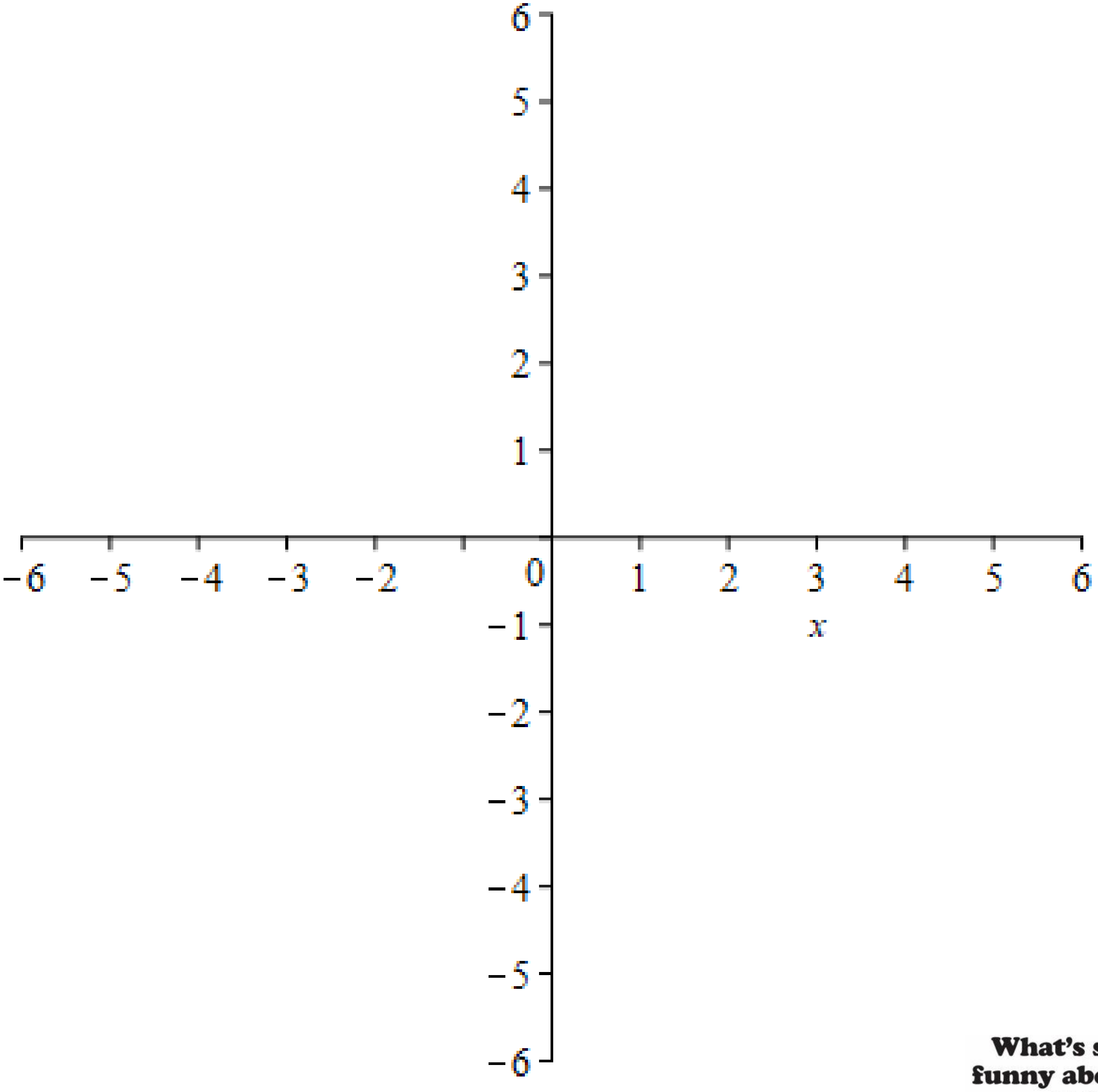
S.A.:

Intercepts:

***x*-int(s):**

***y*-int:**

Sign Chart:



4. $f(x)=\dfrac{x^2+5x+6}{x-3}$

Asymptotes:

H.A.:

V.A.:

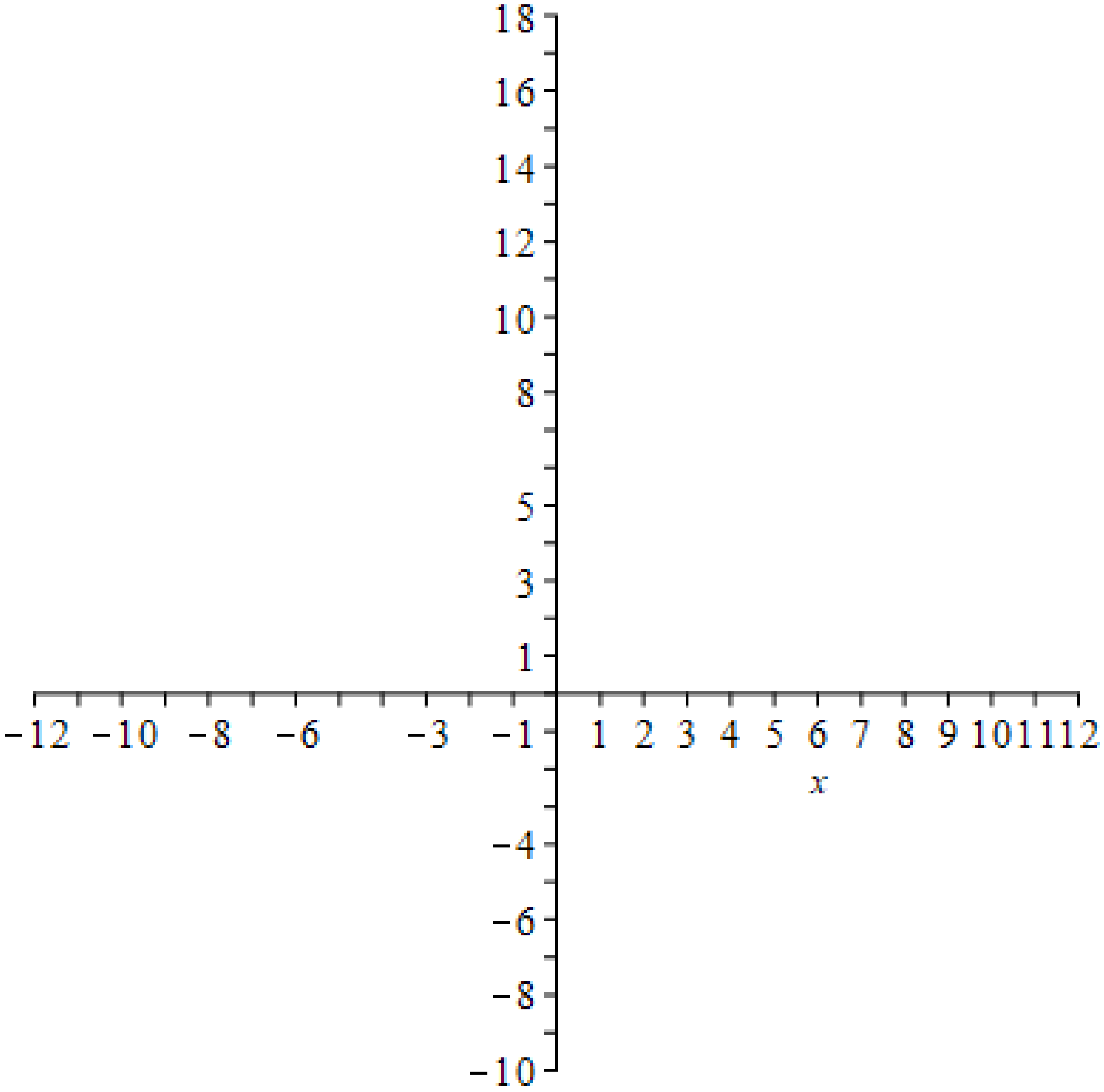
S.A.:

Intercepts:

***x*-int(s):**

***y*-int:**

Sign Chart:



$$\frac{1\cancel{6}\cancel{3}}{\cancel{3}2\cancel{6}}=\frac{1}{2}$$

5. $f(x)=\dfrac{2x^2+3x}{x+1}$

Asymptotes:

H.A.:

V.A.:

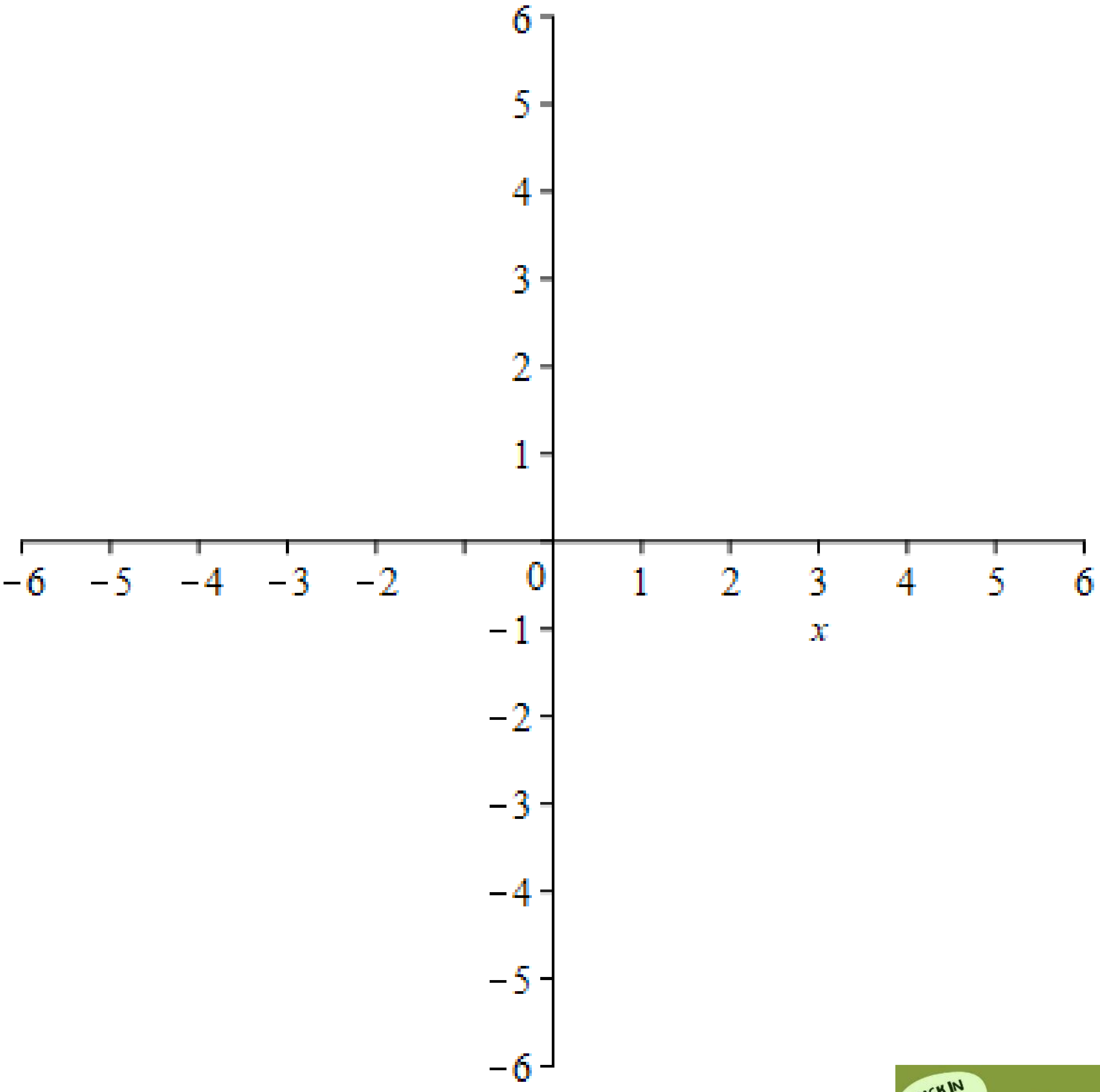
S.A.:

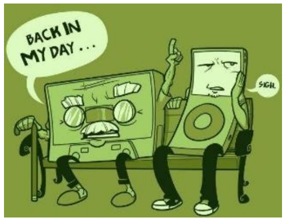
Intercepts:

***x*-int(s):**

***y*-int:**

Sign Chart:





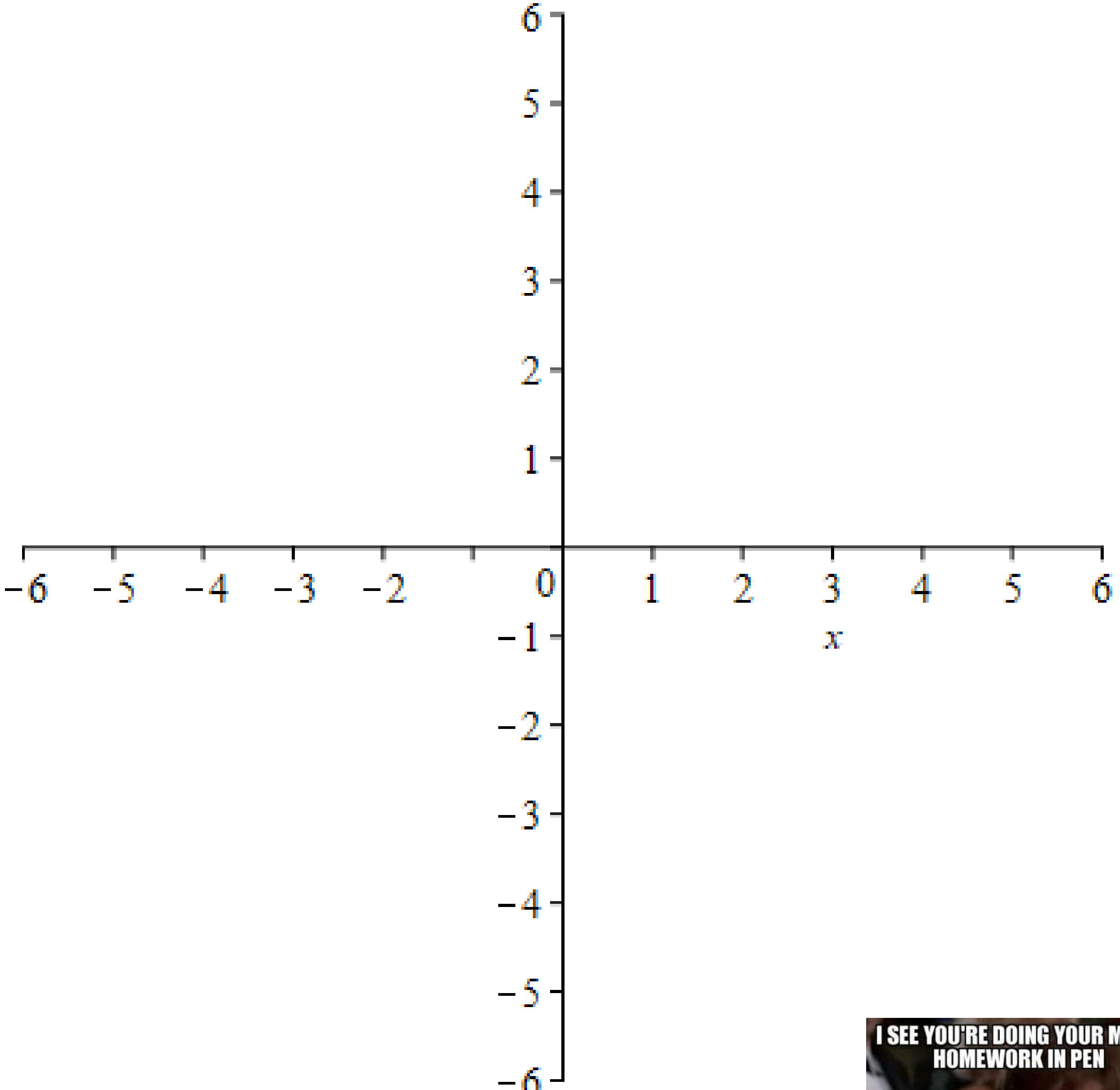
6. $f\left(x\right)=\dfrac{x^3+x^2-2}{x+1}$

Asymptotes:

H.A.:

V.A.:

S.A.:

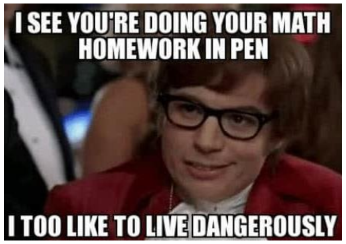


Intercepts:

***x*-int(s):**

***y*-int:**

Sign Chart:



Unreduced Rational Functions:

$f(x)=\dfrac{p(x)}{q(x)}$**, where** $p(x)$ **and** $q(x)$ ***have*** **common real zeros.**

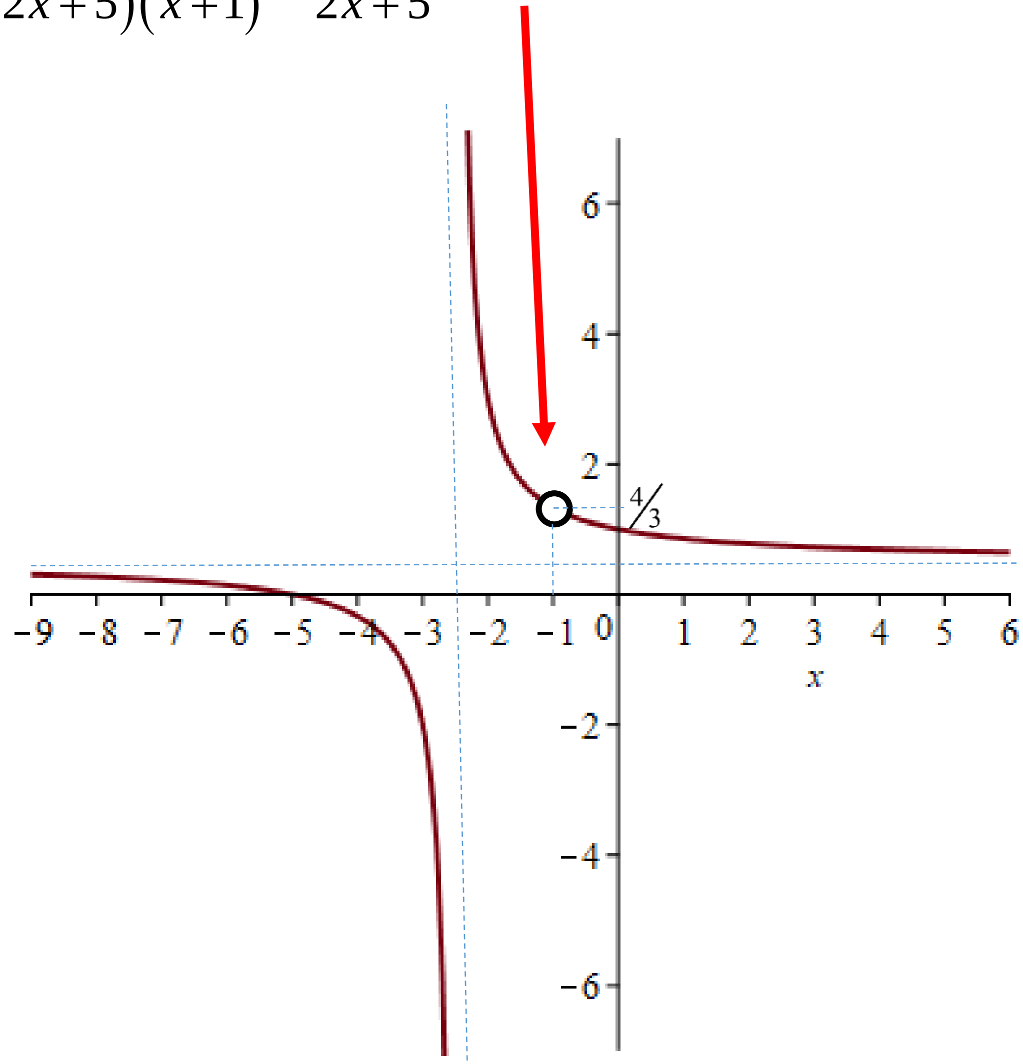
The real zeros of $q(x)$ **are numbers that are not in the domain of** $f(x)$**.**

The first step is to cancel the common factors corresponding to the common real zeros, and analyze the remaining reduced rational function with a restricted domain. ***Again, if they have common imaginary zeros, then the corresponding common factors should be cancelled out, as well.***

> DON'T CONFUSE AN ASYMPTOTE WITH A HOLE IN THE GRAPH!
>
> D. EGLEY

Examples:

1. $f(x)=\dfrac{x^2+6x+5}{2x^2+7x+5}=\dfrac{(x+1)(x+5)}{(2x+5)(x+1)}=\dfrac{x+5}{2x+5};x\neq -1$



$$\textbf{2.}\quad f(x)=\frac{x^2+x-30}{x+6}=\frac{(x+6)(x-5)}{x+6}=x-5;\, x\neq -6$$