***Linear Factorization Theorem:***

**Every polynomial function** $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ **of degree** $n \geq 1$ **can be factored into** $f(x) = a_n(x - c_1)(x - c_2)\cdots(x - c_n)$ **, where** $c_1, c_2, \ldots, c_n$ **are the *n* zeros of** $f(x)$**.**

**Examples:**

**1. Write the polynomial function** $f(x) = x^3 + 3x^2 - x - 3$ **in linear factored form.**

$$x^3 + 3x^2 - x - 3 = x^2(x+3) - (x+3) = (x+3)(x^2 - 1)$$

So $f(x) = x^3 + 3x^2 - x - 3 = \boxed{(x+3)(x-1)(x+1)}$

**2. Write the polynomial function $f(x) = 2x^3 + 2x$ in linear factored form.**

$$2x^3 + 2x = 2x(x^2 + 1)$$

So $f(x) = 2x^3 + 2x = \boxed{2x(x-i)(x+i)}$

**3. Find a polynomial function, $f(x)$, of degree 3 with zeros of $-2$, 3, and 5 and with $f(0) = 60$.**

$$f(x) = a_3(x+2)(x-3)(x-5)$$

$$\Rightarrow 60 = a_3(0+2)(0-3)(0-5) \Rightarrow 60 = 30a_3 \Rightarrow a_3 = 2$$

So $\boxed{f(x) = 2(x+2)(x-3)(x-5)}$

**4. Find a polynomial function with real coefficients, $f(x)$, of degree 3 with zeros of 2 and *i* and with $f(0)=6$.**

$$f(x)=a_3(x-2)(x-i)(x+i)=a_3(x-2)(x^2+1)$$

$$\Rightarrow 6=a_3(0-2)(0^2+1)\Rightarrow 6=-2a_3\Rightarrow a_3=-3$$

So $\boxed{f(x)=-3(x-2)(x^2+1)}$

**5. Find a polynomial function with real coefficients, $f(x)$, of degree 3 with zeros of 1 and $1+i$ and with $f(0)=-2$.**

$$f(x)=a_3(x-1)\left[x-(1+i)\right]\left[x-(1-i)\right]=a_3(x-1)\left[(x-1)-i\right]\left[(x-1)+i\right]$$

$$=a_3(x-1)\left[(x-1)^2+1\right]=a_3(x-1)(x^2-2x+2)$$

$$\Rightarrow -2=a_3(0-1)(0^2-2\cdot 0+2)\Rightarrow -2=-2a_3\Rightarrow a_3=1$$

So $\boxed{f(x)=(x-1)(x^2-2x+2)}$

***Rational Functions:***

$$f(x)=\frac{p(x)}{q(x)}$$, **where** $p(x)$ **and** $q(x)$ **are polynomial functions with no common real zeros.**

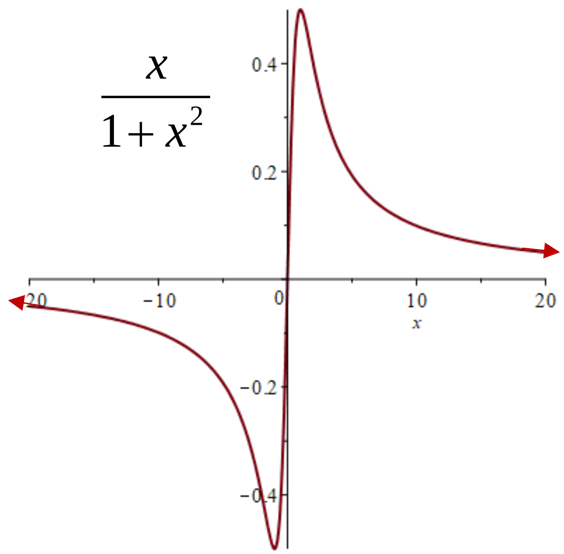
***Asymptotes:*** **(Polynomials don't have them.)**

**Lines that the graph of a rational function approach.**

**Horizontal Asymptote: (End Behavior)**

**If the degree of** $p(x)$ **is less than the degree of** $q(x)$**, then** $y=0$ **(*x*-axis) is the horizontal asymptote.**

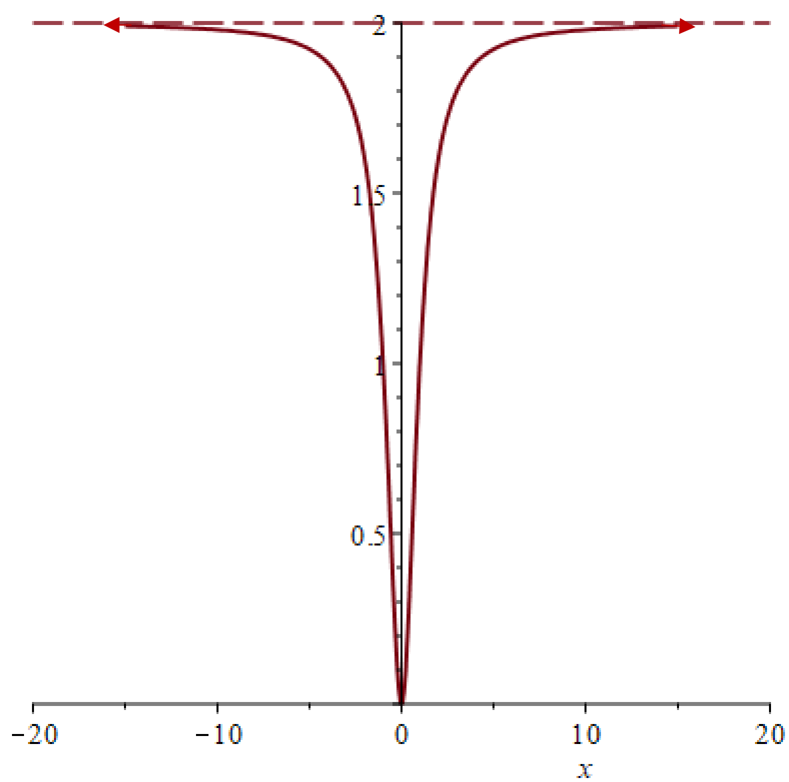
$$\frac{x}{1+x^2}$$



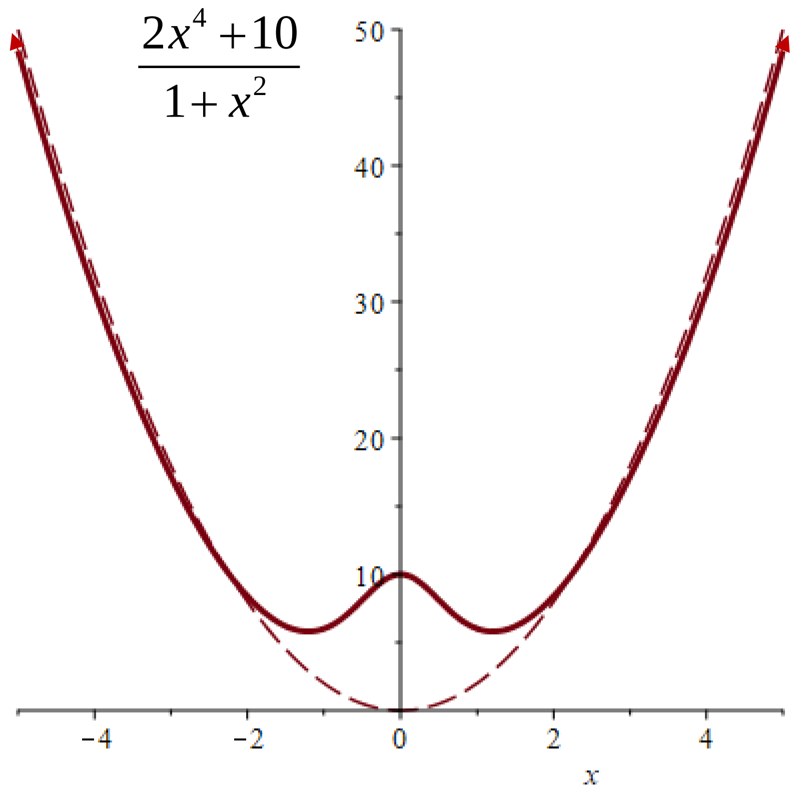
**If the degree of $p(x)$ is equal to the degree of $q(x)$, then**

$$y = \frac{\text{leading coefficient of } p(x)}{\text{leading coefficient of } q(x)}$$ **is the horizontal asymptote.**

$$\frac{2x^2}{1+x^2}$$



**If the degree of $p(x)$ is greater than the degree of $q(x)$, then there is no horizontal asymptote. The end behavior is like the polynomial function formed from the ratio of the leading terms of $p(x)$ and $q(x)$.**




**The end behavior is like** $\frac{2x^4}{x^2} = 2x^2$.

**Examples: Determine the horizontal asymptote/end behavior of the following rational functions.**

**1.** $f(x)=\dfrac{2x^3+x-1}{3x^3+x^2}$

$$y=\frac{2}{3}$$

**2.** $f(x)=\dfrac{2x^2+x-1}{3x^3+x^2}$

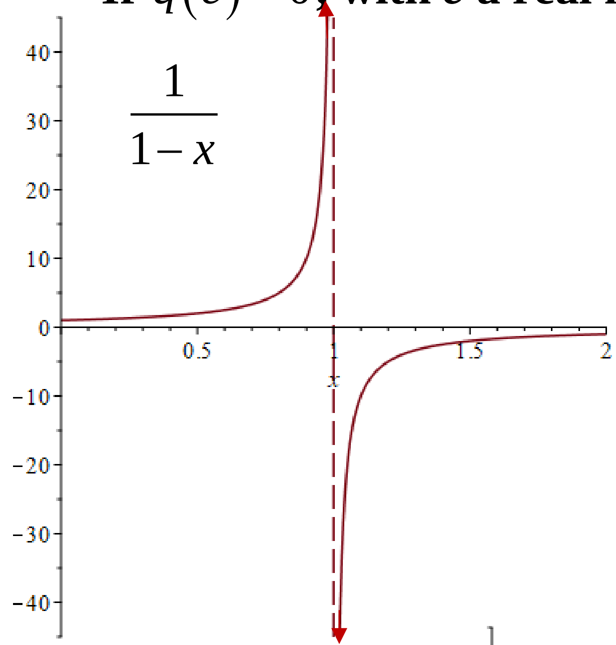
$$y=0$$

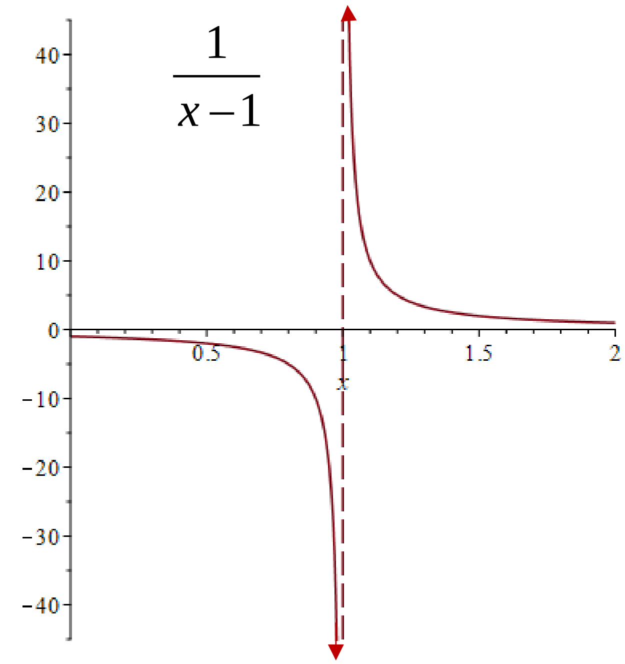
**3.** $f(x)=\dfrac{2x^5+x-1}{3x^3+x^2}$

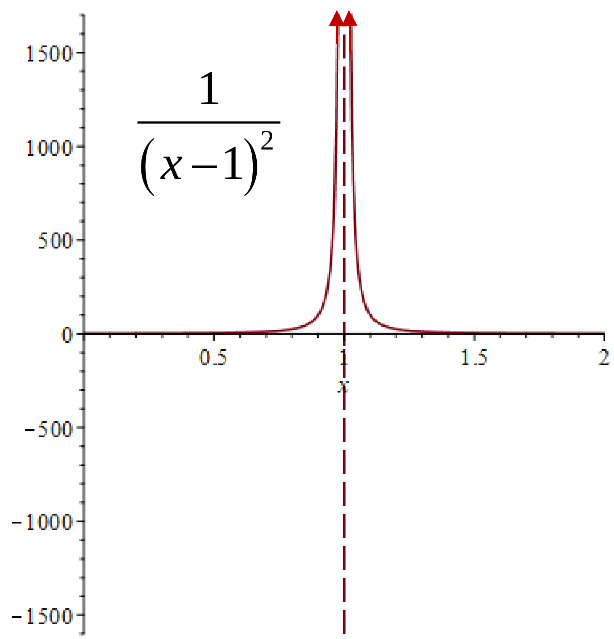
There is no horizontal asymptote, but the end behavior is the same as $y=\dfrac{2}{3}x^2$.

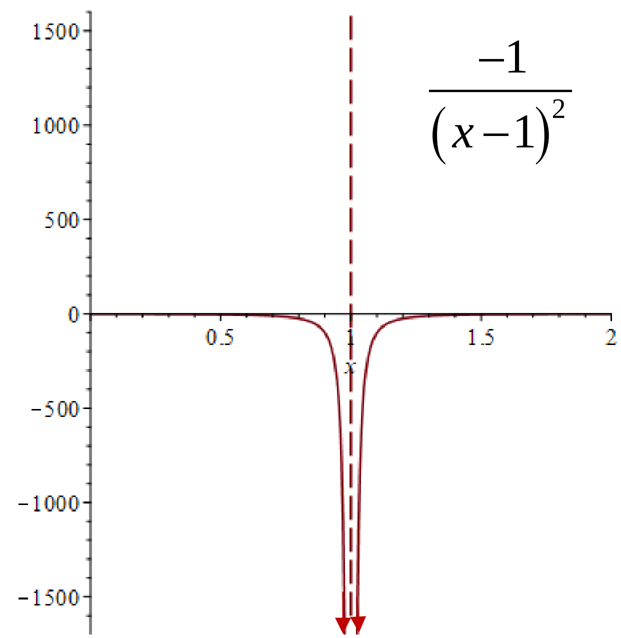
**<u>Vertical Asymptote:</u> (Upward/downward explosions)**

**If $q(c)=0$, with *c* a real number, then $x=c$ is a vertical asymptote.**



**Examples: Determine the vertical asymptotes/explosive behavior of the following rational functions.**

**1.** $f(x) = \dfrac{3x+5}{x-6}$

$$x = 6$$

**2.** $f(x) = \dfrac{x^3+1}{x^2-5x-14}$

$$x^2 - 5x - 14 = (x-7)(x+2)$$

So the vertical asymptotes are $\boxed{x = 7 \text{ and } x = -2}$.

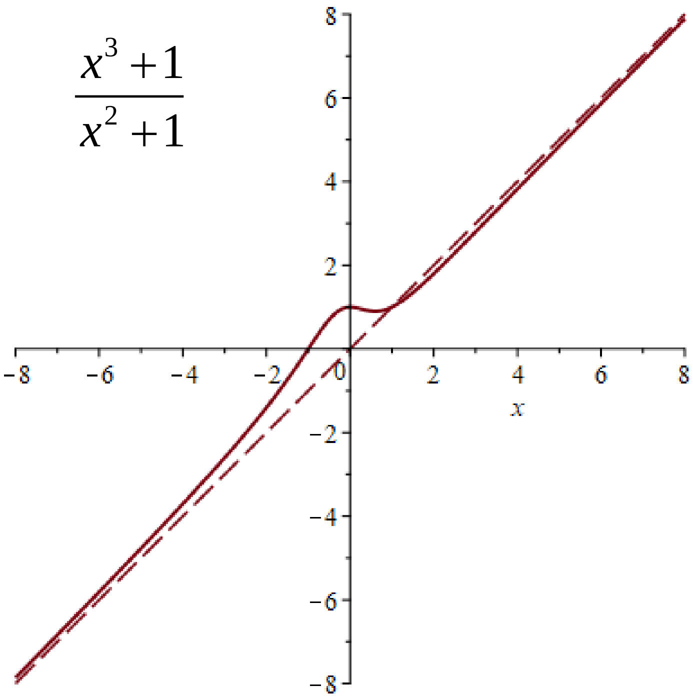
**3.** $f(x) = \dfrac{x^3}{x^4-1}$

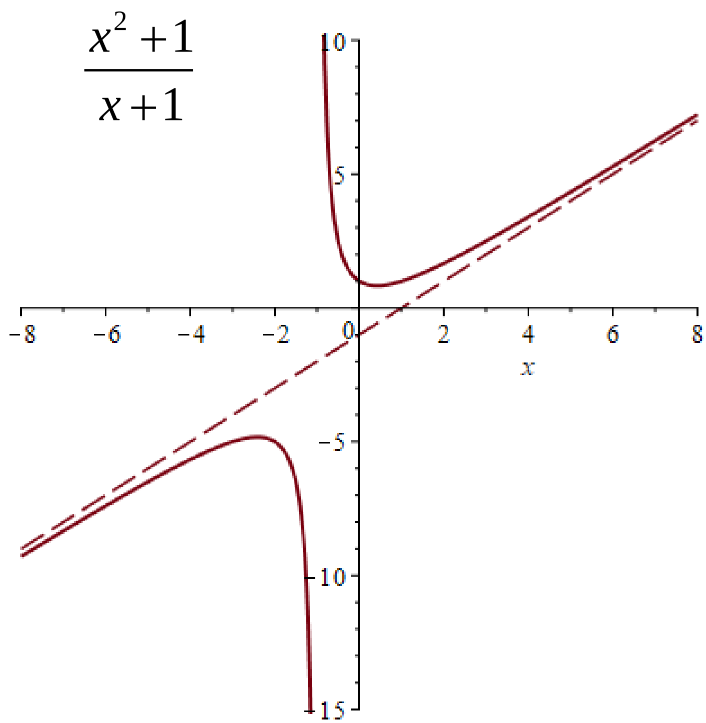
$$x^4 - 1 = (x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1)$$

So the vertical asymptotes are $\boxed{x = 1 \text{ and } x = -1}$.

***Slant/Oblique Asymptote:***

**If the degree of $p(x)$ is 1 more than the degree of $q(x)$, then the rational function will have a slant/oblique asymptote. The equation of the slant asymptote can be determined using polynomial long division.**



**Examples:**

**1.** $f(x)=\dfrac{2x^2+x-1}{x+2}$

$$\begin{array}{r|l} & 2x-3 \\ \hline x+2 & 2x^2+x-1 \\ & -\left(2x^2+4x\right) \\ \hline & -3x-1 \\ & -\left(-3x-6\right) \\ \hline & \qquad\quad 5 \end{array}$$

So $f(x)=\dfrac{2x^2+x-1}{x+2}=2x-3+\dfrac{5}{x+2}$

$\Rightarrow$ The slant asymptote is $\boxed{y=2x-3}$.

**2.** $f(x)=\dfrac{x^3+8}{x^2-5x+6}$

$$
\begin{array}{r|l}
 & x+5 \\
\hline
x^2-5x+6 & x^3+8 \\
 & -\left(x^3-5x^2+6x\right) \\
\cline{2-2}
 & 5x^2-6x+8 \\
 & -\left(5x^2-25x+30\right) \\
\cline{2-2}
 & 19x-22
\end{array}
$$

So $f(x)=\dfrac{x^3+8}{x^2-5x+6}=x+5+\dfrac{19x-22}{x^2-5x+6}$

$\Rightarrow$ The slant asymptote is $\boxed{y=x+5}$.