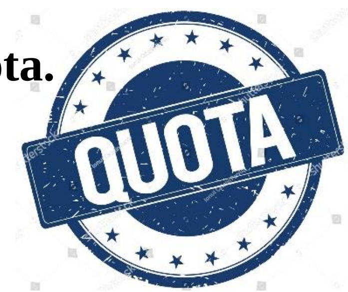
Fairness and Paradoxes in Apportionment Methods:

Fairness: **The only fairness criterion for apportionment methods is that a group should either receive its lower quota or its upper quota of items. An apportionment method that guarantees this is said to satisfy quota.**



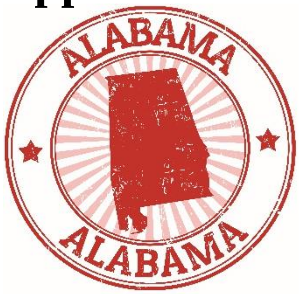
The only apportionment method that we've discussed that always satisfies quota is Hamilton's Method.

So Hamilton's Method is the only fair apportionment method.

Paradoxes: **In the history of using Hamilton's Method, some strange things have occurred that have been called apportionment paradoxes.**

The Alabama Paradox:

The total number of items to be allocated is increased, but a group is apportioned fewer items.



Example:

State	A	B	C	D	Total
Population	504	456	404	61	1,425

57 items are to be apportioned. The standard divisor is $\frac{1,425}{57} = 25$**.**

State	A	B	C	D	Total
Population	504	456	404	61	1,425
Standard Quota	20.16	18.24	16.16	2.44	57
Lower Quota	20	18	16	2	56
Hamilton Apportionment	20	18	16	3	57

Suppose that the number of items to be allocated increases from 57 to 58.

The new standard divisor is $\frac{1,425}{58} = 24.568...$**.**

State	A	B	C	D	Total
Population	504	456	404	61	1,425
Standard Quota	20.51	18.56	16.44	2.48	58
Lower Quota	20	18	16	2	56
Hamilton Apportionment	21	19	16	2	58

Did the Alabama Paradox occur?

Yes, the number of items to be allocated increased from 57 to 58, but state D went from getting 3 to only getting 2 of them.

The Population Paradox:

One group loses items to another group even though the population of the first group grew at a higher percentage than the second group.

Example:

State	A	B	C	Total
Original Population	**53**	**99**	**224**	**376**
New Population	**68**	**125**	**257**	**450**
Percent Change in Population	$\frac{68-53}{53} = .2830$ $= 28.30\%$	$\frac{125-99}{99} = .2626$ $= 26.26\%$	$\frac{257-224}{224} = .1473$ $= 14.73\%$	

The number of items to be allocated is 24.

Original Hamilton:

The standard divisor is $\frac{376}{24} = 15.\overline{6}$**.**

State	A	B	C	Total
Original Population	53	99	224	376
Standard Quota	3.38	6.32	14.30	24
Lower Quota	3	6	14	23
Hamilton Apportionment	4	6	14	24

New Hamilton:

The standard divisor is $\frac{450}{24} = 18.75$**.**

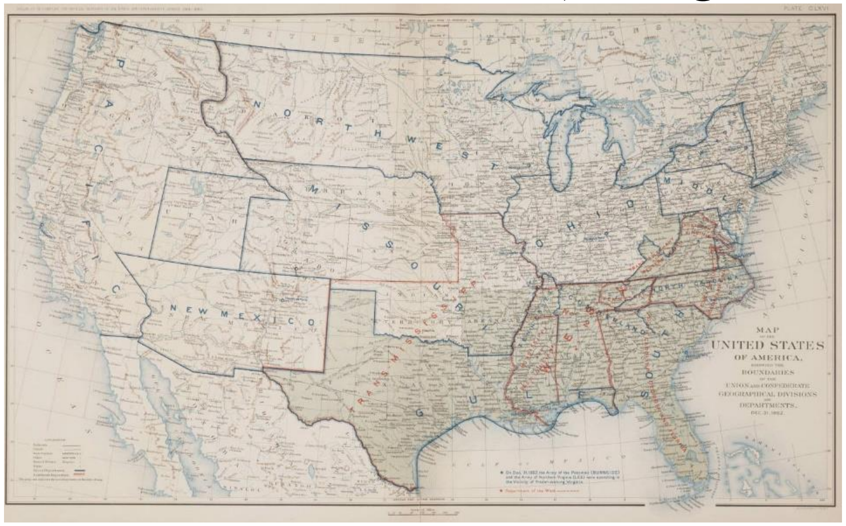
State	A	B	C	Total
New Population	68	125	257	450
Standard Quota	3.63	6.67	13.71	24
Lower Quota	3	6	13	22
Hamilton Apportionment	3	7	14	24

Did the Population Paradox occur?

Yes, State A lost an item to State B, even though State A had a larger percent increase in its population than State B did.

The New States Paradox:

The addition of a new group(and a proportional number of new items) changes the apportionments of the original groups.



Example:

State	A	B	C	Total
Population	209	769	2,022	3,000

60 items are to be allocated. The standard divisor is $\dfrac{3{,}000}{60} = 50$**.**

State	A	B	C	Total
Population	209	769	2,022	3,000
Standard Quota	4.18	15.38	40.44	60
Lower Quota	4	15	40	59
Original Hamilton Apportionment	4	15	41	60

A new state, D, is added with a population of 260, and a proportional number of items, 5, is also added.

State	A	B	C	D	Total
Population	209	769	2,022	260	3,260

The new standard divisor is $\frac{3260}{65} = 50.1538$**....**

State	A	B	C	D	Total
Population	209	769	2,022	260	3,260
Standard Quota	4.17	15.33	40.32	5.18	65
Lower Quota	4	15	40	5	64
New Hamilton Apportionment	4	16	40	5	65

Did the New States Paradox occur?

Yes, the addition of State D changed the original apportionment for the original states A, B, and C.

One last Example:

A country has five states, and its house of representatives is apportioned by the Hamilton Method. Complete the apportionments for house sizes of 81 and 82.

State	A	B	C	D	E	Total
Population	5,576,330	1,387,342	3,334,241	7,512,860	310,968	18,121,741
Original Quotas	24.925	6.201	14.903	33.581	1.390	81
Original Hamilton	25	6	15	34	1	81
New Quotas	25.233	6.278	15.087	33.995	1.407	82
New Hamilton	25	6	15	34	2	82

Did the Alabama Paradox Occur?

No. When the house size increased from 81 to 82, none of the states lost a seat.

Balinski and Young's Imposibility Theorem(1980):

Any apportionment method that doesn't violate the quota rule must produce paradoxes, and any apportionment rule that doesn't produce paradoxes must violate the quota rule.