Rational Zero/Root Theorem:

For $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ **a polynomial function with integer coefficients, if the reduced rational number** $\frac{p}{q}$ **is a zero of** $f(x)$**, then *p* must be a factor of** a_0 **and *q* must be a factor of** a_n**.**

Examples: List all the possible rational zeros of the following polynomial functions.

1. $f(x)=x^5-3x^2+1$

Factors of the constant term, 1: ±1

Factors of the leading coefficient, 1: ±1

Possible rational zeros: $\boxed{\pm1}$

2. $f(x)=10x^5+3x^4-2x^2-6$

Factors of the constant term, -6: $\pm1,\pm2,\pm3,\pm6$

Factors of the leading coefficient, 10: $\pm1,\pm2,\pm5,\pm10$

Possible rational zeros: $\boxed{\pm1,\pm\frac{1}{2},\pm\frac{1}{5},\pm\frac{1}{10},\pm2,\pm\frac{2}{5},\pm3,\pm\frac{3}{2},\pm\frac{3}{5},\pm\frac{3}{10},\pm6,\pm\frac{6}{5}}$

The Fundamental Theorem of Algebra:

Every polynomial function of degree $n \geq 1$ has *n* zeros in the set of complex numbers. Repeated zeros are counted separately.

If the coefficients in the polynomial function are real numbers, then the imaginary zeros occur in conjugate pairs, i.e. if $a+bi$ is a zero, then so is $a-bi$.

Descartes' Rule of Signs:

For $f(x)=a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ a polynomial function with real coefficients, then

1. **The number of positive zeros is either the number of sign changes in the coefficients of $f(x)$, or less than the number of sign changes by an even number.**

2. **The number of negative zeros is either the number of sign changes in the coefficients of $f(-x)$, or less than the number of sign changes by an even number.**

Examples: Find the possible numbers of positive zeros, negative zeros, and imaginary zeros for the following polynomial functions.

1. $f\left(x\right)=x^5-3x^2+1$

coefficients of *f*(*x*)	**1**	**-3**	**1**
coefficients of *f*(-*x*)	**-1**	**-3**	**1**

There are two sign changes in the coefficients of the original polynomial, so there are 2 or 0 positive zeros.

$f\left(-x\right)=-x^5-3x^2+1$, so there is one sign change in the coefficients of $f\left(-x\right)$, and therefore there is 1 negative zero.

# of positives	**# of negatives**	**# of imaginaries**
0	**1**	**4**
2	**1**	**2**

2. $f(x) = 10x^5 + 3x^4 + 2x^2 - 6$

coefficients of *f*(*x*)	**10**	**3**	**2**	**-6**
coefficients of *f*(-*x*)	**-10**	**3**	**2**	**-6**

There is one sign change in the coefficients of the original polynomial, so there is 1 positive zero.
$f(-x) = -10x^5 + 3x^4 + 2x^2 - 6$, so there are two sign changes in the coefficients of $f(-x)$, and therefore there are 2 or 0 negative zeros.

# of positives	**# of negatives**	**# of imaginaries**
1	**2**	**2**
1	**0**	**4**

Find all the zeros of the following polynomial functions:

1. $f(x)=x^3-x^2+x-1$ **{Factor, if possible.}**

$$x^3-x^2+x-1=x^2(x-1)+(x-1)=(x-1)(x^2+1)$$

So the zeros are $\boxed{1,i,-i}$.

Descartes' indicates 3 or 1 positive zeros, 0 negative zeros, and 0 or 2 imaginary zeros.

2. $f(x)=x^3-3x+2$ **{Check easy values like ±1.}**

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -3 & 2 \\ & & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & \boxed{0} \end{array}$$

$$x^3-3x+2=(x-1)(x^2+x-2)=(x-1)(x+2)(x-1)=(x-1)^2(x+2)$$

So the zeros are $\boxed{1,1,-2}$.

Descartes' indicates 2 or 0 positive zeros, 1 negative zero, and 0 or 2 imaginary zeros.

3. $f\left(x\right)=3x^4-4x^3+x^2+6x-2$ **{Check easy values like ±1.}**

$$\begin{array}{r|rrrrr} -1 & 3 & -4 & 1 & 6 & -2 \\ & & -3 & 7 & -8 & 2 \\ \hline & 3 & -7 & 8 & -2 & \boxed{0} \end{array}$$

$3x^4-4x^3+x^2+6x-2=\left(x+1\right)\left(3x^3-7x^2+8x-2\right)$

Remaining possible rational zeros: $\pm1,\pm2,\pm\frac{1}{3},\pm\frac{2}{3}$

Descartes' indicates 3 or 1 remaining positive zeros, no remaining negative zeros, and 0 or 2 remaining imaginary zeros.

$$\begin{array}{r|rrrr} \frac{1}{3} & 3 & -7 & 8 & -2 \\ & & 1 & -2 & 2 \\ \hline & 3 & -6 & 6 & \boxed{0} \end{array}$$

$3x^4-4x^3+x^2+6x-2=\left(x+1\right)\left(x-\frac{1}{3}\right)\left(3x^2-6x+6\right)=3\left(x+1\right)\left(x-\frac{1}{3}\right)\left(x^2-2x+2\right)$

Quadratic formula applied to $x^2-2x+2=0$ leads to $\dfrac{2\pm\sqrt{4-8}}{2}=\dfrac{2\pm2i}{2}=1\pm i$.

So the zeros are $\boxed{-1,\frac{1}{3},1\pm i}$.

4. $f\left(x\right)=x^3-4x^2+2x+4$

Possible rational zeros: $\pm1,\pm2,\pm4$

Descartes' indicates 2 or 0 positive zeros, 1 negative zero, and 0 or 2 imaginary zeros.

$$\begin{array}{r|rrrr} 2 & 1 & -4 & 2 & 4 \\ & & 2 & -4 & -4 \\ \hline & 1 & -2 & -2 & \boxed{0} \end{array}$$

$x^3-4x^2+2x+4=\left(x-2\right)\left(x^2-2x-2\right)$

Quadratic formula applied to $x^2-2x-2=0$ leads to $\dfrac{2\pm\sqrt{4+8}}{2}=\dfrac{2\pm2\sqrt{3}}{2}=1\pm\sqrt{3}$.

So the zeros are $\boxed{2,1\pm\sqrt{3}}$.