Counting Methods:

Example:

A pen has tip options of regular tip, fine tip, or micro tip, and it has ink color options of red ink or green ink. How many different pens are possible?



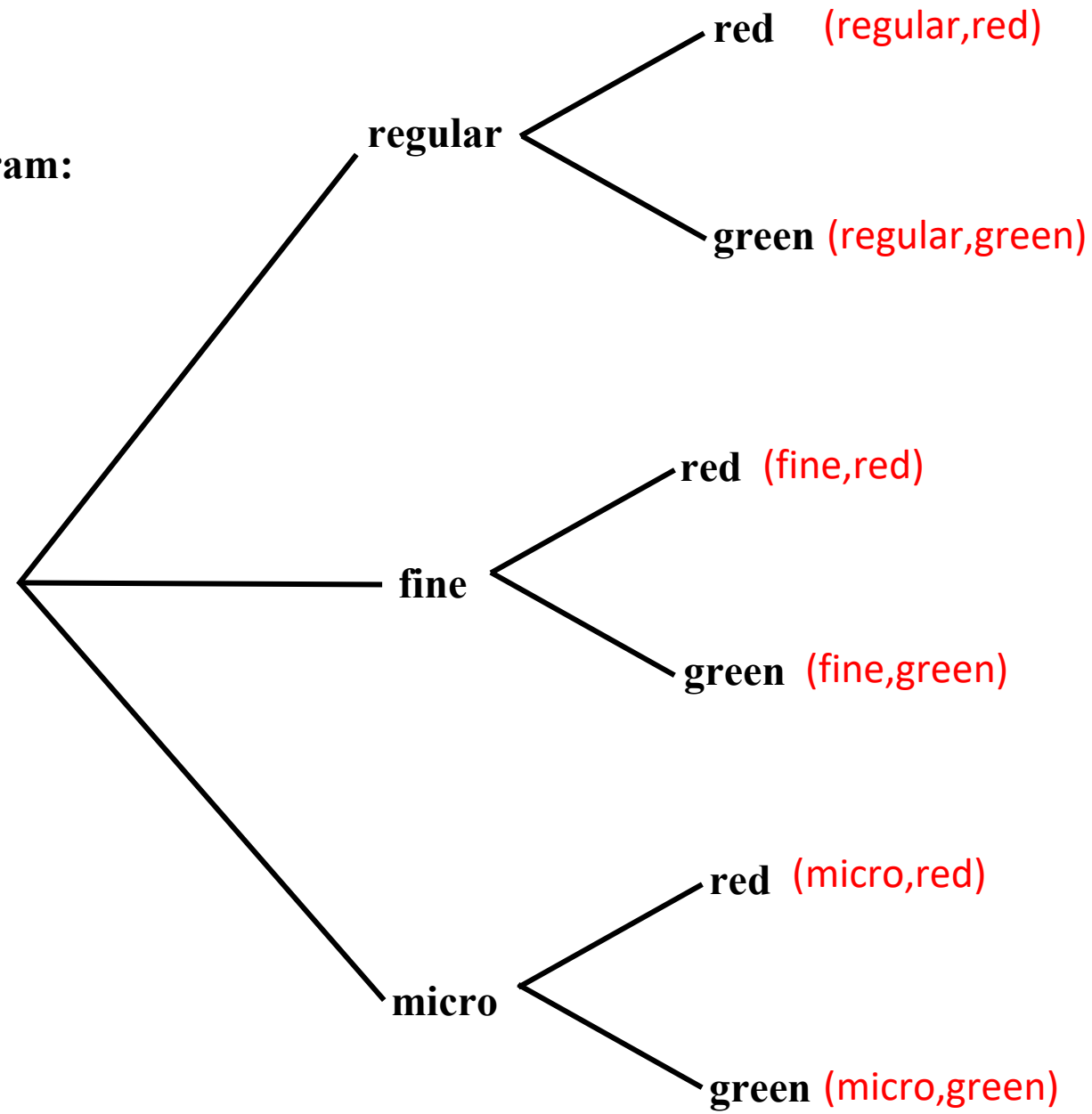
Using a table:

	regular	fine	micro
red	(red,regular)	(red,fine)	(red,micro)
green	(green,regular)	(green,fine)	(green,micro)

The number of pens possible is the number of cells in the table: $3 \times 2 = 6$.

Using a Tree Diagram:





The number of pens possible is the number of branch tips on the right: $3 \times 2 = 6$**.**

The Fundamental Counting Principle:

If a sequence of decisions is to be made, then the number of different ways of making all the decisions is the product of the number of options for each decision.

Examples:

1. A meal consists of 1 of 8 appetizers, 1 of 10 entrees, and 1 of 5 desserts. How many different meals are possible?

$$\underset{\text{appetizer}}{8} \cdot \underset{\text{entree}}{10} \cdot \underset{\text{dessert}}{5} = \boxed{400}$$



2. In a race with 5 horses, how many different first, second, and third place finishes are possible?

$$\underset{\text{first}}{5} \cdot \underset{\text{second}}{4} \cdot \underset{\text{third}}{3} = \boxed{60}$$



3. In a certain small state, license plates consist of three letters followed by two digits.



a) How many different plates are possible?

$$\underset{\text{first}}{26} \cdot \underset{\text{second}}{26} \cdot \underset{\text{third}}{26} \cdot \underset{\text{fourth}}{10} \cdot \underset{\text{fifth}}{10} = \boxed{1{,}757{,}600}$$

b) How many if letters can't repeat?

$$\underset{\text{first}}{26} \cdot \underset{\text{second}}{25} \cdot \underset{\text{third}}{24} \cdot \underset{\text{fourth}}{10} \cdot \underset{\text{fifth}}{10} = \boxed{1{,}560{,}000}$$

c) How many if digits can't repeat?

$$\underset{\text{first}}{26} \cdot \underset{\text{second}}{26} \cdot \underset{\text{third}}{26} \cdot \underset{\text{fourth}}{10} \cdot \underset{\text{fifth}}{9} = \boxed{1{,}581{,}840}$$

d) How many if no repeats?

$$\underset{\text{first}}{26} \cdot \underset{\text{second}}{25} \cdot \underset{\text{third}}{24} \cdot \underset{\text{fourth}}{10} \cdot \underset{\text{fifth}}{9} = \boxed{1{,}404{,}000}$$

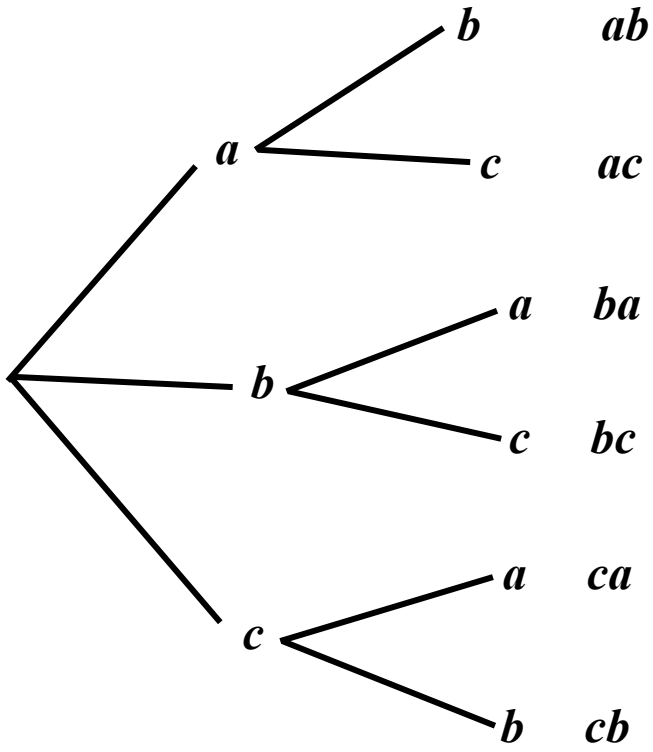
Permutations:

A permutation is an arrangement of objects in a particular order.



Example:

Find all the permutations of the objects $\{a,b,c\}$ of size 2.



There are 6 permutations of size 2 from the 3 objects. If we just wanted to know how many, we could have used the Fundamental Counting Principle.

$$\underset{\text{first}}{3} \cdot \underset{\text{second}}{2} = 6$$

In general, the number of permutations of size *r* from *n* objects is abbreviated as $_nP_r$. So far, we know that $_3P_2 = 6$. There's a nice formula for the value of $_nP_r$ in general, but it involves things called factorials.

Factorials:

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1 \text{ or } n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n$$

So $1! = 1$**.**

$$2! = 2 \cdot 1 = 2$$

$$3! = 3 \cdot 2 \cdot 1 = 6$$

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 4 \cdot 3! = 4 \cdot 6 = 24$$

$$5! = 5 \cdot 4! = 5 \cdot 24 = 120$$

$$6! = 6 \cdot 5! = 6 \cdot 120 = 720$$

By special definition, $0! = 1$**.**

Most scientific calculators have a key for factorials: *n!* or *x!; check out the manual.*

$${}_nP_r = \frac{n!}{(n-r)!}$$ Most scientific calculators have a key for the number of permutations.

Let's check it out for ${}_3P_2$, which we already know is equal to 6.

$${}_3P_2 = \frac{3!}{(3-2)!} = \frac{3!}{1!} = \frac{6}{1} = 6$$

Examples:

1. Five solo singers are to perform their acts at a nightclub on Saturday night. How many different orders of their appearances are possible?



$${}_5P_5 = \frac{5!}{(5-5)!} = \frac{5!}{0!} = \frac{120}{1} = \boxed{120}$$

Or

Fundamental Counting Principle

$$\underset{\text{first}}{5} \cdot \underset{\text{second}}{4} \cdot \underset{\text{third}}{3} \cdot \underset{\text{fourth}}{2} \cdot \underset{\text{fifth}}{1} = \boxed{120}$$

2. From a group of 7 people, a president, vice-president, and secretary will be selected, how many different selections are possible?



$$_7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = \frac{5{,}040}{24} = \boxed{210}$$

Or

Fundamental Counting Principle

$$\underset{\text{president}}{7} \cdot \underset{\text{vice-president}}{6} \cdot \underset{\text{secretary}}{5} = \boxed{210}$$

3. In a race with 8 horses, how many different first, second, and third place finishes are possible?



$$_8P_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = 8 \cdot 7 \cdot 6 = \boxed{336}$$

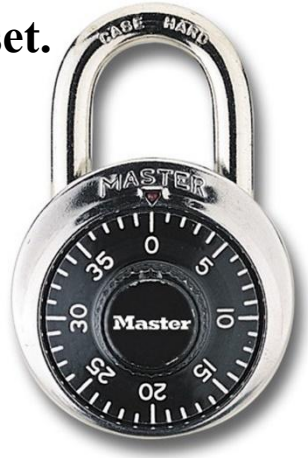
Or

Fundamental Counting Principle

$$\underset{\text{first}}{8} \cdot \underset{\text{second}}{7} \cdot \underset{\text{third}}{6} = \boxed{336}$$

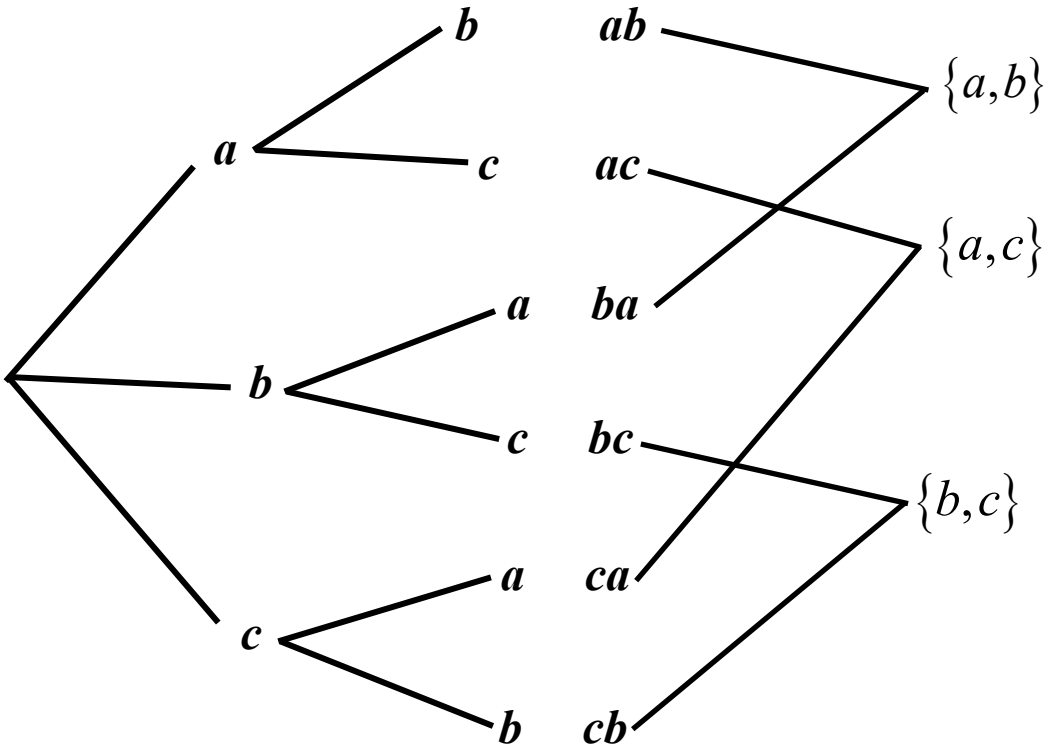
Combinations:

A combination is a selection of objects without regard to order, i.e. a subset.



Example:

Find all the combinations of the objects $\{a,b,c\}$ of size 2.



There are 6 permutations of size 2 from the 3 objects, but only 3 combinations of size 2 from the 3 objects.

In general, the number of combinations of size *r* from *n* objects is abbreviated as $_nC_r$. **So far, we know that** $_3C_2 = 3$, **and** $_3C_2 = 3 = \frac{6}{2} = \frac{_3P_2}{2!}$. **This is true in general, and leads to a nice formula for** $_nC_r$. $_nC_r = \frac{n!}{r!\cdot(n-r)!}$

Most scientific calculators have a key for determining the number of combinations.

Examples:

1. A three-person committee is to be selected from a group of 12 people. How many different committees are possible?

$$_{12}C_3 = \frac{12!}{3!\cdot 9!} = \frac{12\cdot 11\cdot 10}{3\cdot 2\cdot 1} = \boxed{220}$$



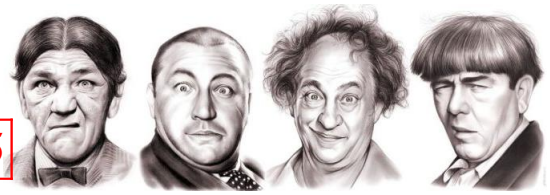
2. In a certain lottery, you must select 6 numbers from the numbers 1-50. How many different lottery selections are possible?

$$_{50}C_6 = \frac{50!}{6!\cdot 44!} = \frac{50\cdot 49\cdot 48\cdot 47\cdot 46\cdot 45}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = \boxed{15{,}890{,}700}$$



3. A group consists of 7 men and 8 women. A committee of 4 people will be selected.

a) How many different 4-person committees are possible?



$$_{15}C_4 = \frac{15!}{4!\cdot 11!} = \frac{15\cdot 14\cdot 13\cdot 12}{4\cdot 3\cdot 2\cdot 1} = \boxed{1{,}365}$$

b) How many different 4-person committees consisting of 4 women are possible?

$$_{8}C_4 = \frac{8!}{4!\cdot 4!} = \frac{8\cdot 7\cdot 6\cdot 5}{4\cdot 3\cdot 2\cdot 1} = \boxed{70}$$

c) How many different 4-person committees consisting of 3 women and 1 man are possible?

$_{8}C_3 = 56$	$_{7}C_1 = 7$
Which 3 women?	**Which 1 man?**

$$56\cdot 7 = \boxed{392}$$

d) How many different 4-person committees consisting of 2 women and 2 men are possible?

$_8C_2 = 28$	$_7C_2 = 21$
Which 2 women?	**Which 2 men?**

$28 \cdot 21 = \boxed{588}$

e) How many different 4-person committees have at least 1 man?

= the number of committees with at least one man

= the total number of committees – the number of committees with no men

$= 1{,}365 - 70 = \boxed{1{,}295}$

Sometimes it's easier to calculate quantities indirectly!

Finding Probabilities Using Counting Techniques:

In the case of the equally likely assumption, $P(E)=\dfrac{n(E)}{n(S)}$.

Examples:

1. Allen, Bob, Carl, and David will be seated in a row of 4 chairs at random.



a) How many different ways can they be seated?

$$n(S)=\underset{\text{first}}{4}\cdot\underset{\text{second}}{3}\cdot\underset{\text{third}}{2}\cdot\underset{\text{fourth}}{1}=\boxed{24}$$

b) How many different ways can Carl be seated in the first chair and Allen in the fourth chair?

$$n(E)=\underset{\text{first}}{1}\cdot\underset{\text{second}}{2}\cdot\underset{\text{third}}{1}\cdot\underset{\text{fourth}}{1}=\boxed{2}$$

c) What's the probability that Carl will be seated in the first chair and Allen in the fourth chair?

$$P(E)=\frac{n(E)}{n(S)}=\frac{2}{24}=\frac{1}{12}$$

2. A group consists of 4 men and 5 women. Three people will be selected at random to attend a conference.

a) How many different selections of 3 people from the group are possible?

$$_9C_3 = \boxed{84}$$



b) What's the probability that the 3 people selected are all women?

$$\frac{_5C_3}{_9C_3} = \frac{10}{84} = \frac{5}{42}$$

c) What's the probability that the 3 people selected are all men?

$$\frac{_4C_3}{_9C_3} = \frac{4}{84} = \frac{1}{21}$$

d) What's the probability that the 3 people selected will consist of 2 women and 1 man?

$$\frac{_5C_2 \cdot {}_4C_1}{_9C_3} = \frac{10 \cdot 4}{84} = \frac{40}{84} = \frac{10}{21}$$

3. Three cards will be randomly selected from a 52-card deck without replacement.

a) What's the probability that it will consist of all hearts?



$$\frac{{}_{13}C_3}{{}_{52}C_3} = \frac{286}{22{,}100} = \frac{11}{850}$$

b) What's the probability that it will consist of exactly 2 aces?

$$\frac{{}_4C_2 \cdot {}_{48}C_1}{{}_{52}C_3} = \frac{6 \cdot 48}{22{,}100} = \frac{288}{22{,}100} = \frac{72}{5{,}525}$$

c) What's the probability that it will consist of 2 aces and a king?

$$\frac{{}_4C_2 \cdot {}_4C_1}{{}_{52}C_3} = \frac{6 \cdot 4}{22{,}100} = \frac{24}{22{,}100} = \frac{6}{5{,}525}$$