Last time: 1st derivative test

$f' > 0$ on $I \Rightarrow f$ is increasing

$f' < 0$ on $I \Rightarrow f$ is decreasing

values of x at which $f' = 0$ or undefined

x	
f'	
f	

3.4. Concavity and the second derivative test.

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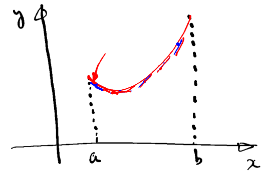

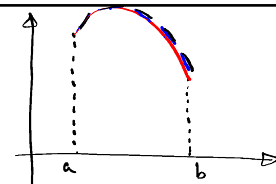

concave up

tangent lines are below graph

Slopes of tangent lines: negative → less negative → 0 → positive → more positive

f' is increasing

$f'' > 0$

$f'' > 0$ on $(a,b) \Rightarrow f'$ increasing $\Rightarrow f$ is concave up

concave down

tangent lines are above graph

positive → less positive → 0 → negative → more negative

f' is decreasing

$f'' < 0 \Rightarrow f'$ is decreasing $\Rightarrow f$ is concave down

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Test for Concavity

(1) $f'' > 0$ on an interval $\Rightarrow f$ is concave up on that interval.

(2) $f'' < 0$ on an interval $\Rightarrow f$ is concave down on that interval.

happy → positive   sad → negative

* Find intervals of concavity of a function.

(1) Find f''. (2) Find values of x at which $f'' = 0$ or f'' is undefined

(3) Make a table

x			
f''	+	−	+
f	∪	∩	∪

E.g. $f(x) = x^5 - 5x + 2$. Determine intervals of concavity.

$f'(x) = 5x^4 - 5$. $f''(x) = 20x^3$

$f'' = 0$ when $\boxed{x = 0}$

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x	~~−2~~	$\boxed{0}$	~~2~~
f''	−	0	+
f	∩		∪

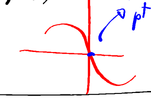
$f''(x) = 20x^3$

inflection point of f

Conclusion: f is concave down on $(-\infty, 0)$

f is concave up on $(0, \infty)$

inflection pt



E.g. $f(x) = \dfrac{2x^2}{3x^2+1}$. Intervals of concavity:

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$f'(x) = \dfrac{4x\cdot(3x^2+1) - 6x\cdot(2x^2)}{(3x^2+1)^2}$ $f(x) = \dfrac{2x^2}{3x^2+1}$

$f'(x) = \dfrac{12x^3 + 4x - 12x^3}{(3x^2+1)^2} = \dfrac{4x}{(3x^2+1)^2}$

$f''(x) = \dfrac{4\cdot(3x^2+1)^2 - 2(3x^2+1)\cdot 6x \cdot 4x}{(3x^2+1)^4}$

$= \dfrac{\cancel{(3x^2+1)}[4\cdot(3x^2+1) - 48x^2]}{(3x^2+1)^{\cancel{4}\,3}} = \dfrac{-36x^2+4}{(3x^2+1)^3}$.

$\boxed{f''(x) = \dfrac{-36x^2+4}{(3x^2+1)^3}}$ < $f'' = 0$; ~~f'' is undefined~~

always > 0

$f'' = 0$ when $-36x^2 + 4 = 0$; $-36x^2 = -4$; $x^2 = \frac{1}{9}$; $\boxed{x = \pm\frac{1}{3}}$

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x	~~−1~~	$-1/3$	~~0~~	$1/3$	
$f''(x)$	−	0	+	0	−
$f(x)$	∩		∪		∩

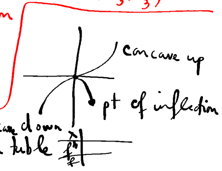
$f''(x) = \dfrac{-36x^2+4}{(3x^2+1)^3}$

Conclusion: f is concave down on $(-\infty, -\frac{1}{3}) \cup (\frac{1}{3}, \infty)$

is concave up on $(-\frac{1}{3}, \frac{1}{3})$.

$x = -\frac{1}{3}$ → point of inflection

$x = \frac{1}{3}$ → point of inflection



pt of inflection can be found from table f''

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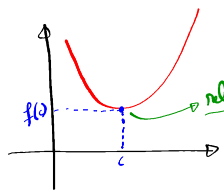
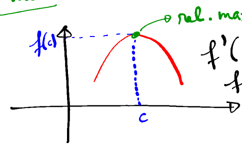
2nd derivative test: another strategy to find relative extrema.



rel. min. $f'(c) = 0$, $f''(c) > 0$



rel. max $f'(c)=0$, $f''(c) < 0$

2nd derivative test.

If $f'(c)=0$ and $f''(c) > 0$, then f has a rel. min at $x=c$

If $f'(c)=0$ and $f''(c) < 0$, then f has a rel. max at $x=c$

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E.g. of using second derivative test to find extrema

$f(x) = -x^4 + 4x^3 + 8x^2$.

Find rel. extrema of f using the second derivative test.

$\boxed{f'(x) = -4x^3 + 12x^2 + 16x}$ $\leftarrow 4$

$f''(x) = \boxed{-12x^2 + 24x + 16}$. $\leftarrow -1$

* $f'(x)=0$. $-4x^3+12x^2+16x = 0$

$-4x(x^2-3x-4)=0$

$-4x(x-4)(x+1)=0$

$x = 0; 4; -1$ ← candidates for rel max/min

Plug $\boxed{x=0}$ into $f''(x)$: $f''(0) = 16 > 0$

$\Rightarrow x=0$ corresponds to a rel. min

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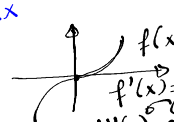
$\boxed{x=4}$ plug into f'': $f''(4) < 0$

→ gives a rel. max

$\boxed{x=-1}$ $f''(-1) < 0$

→ gives a rel. max



$f(x) = x^3$

$f'(x) = 3x^2 = 0$ $\boxed{x=0}$

$f''(x) = 6x = 0$ at $x=0$

$f(x) = x^4$ $f'(x) = 4x^3$; $f''(x) = 12x^2 = 0$ $\boxed{x=0}$

* Note: If $\boxed{f'(c)=0 \text{ and } f''(c)=0}$

then 2nd derivative test fails. (inconclusive)

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