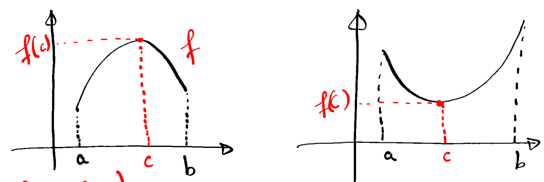
3.1. Finding extrema of a function on a closed interval.

* Find critical numbers of a function.
* Technique to find extrema of a function on a closed interval $[a,b]$.

* Extrema: minima, maxima



$(c, f(c))$ is a maxima of f on $[a,b]$

$(c, f(c))$ is a minima of f on $[a,b]$

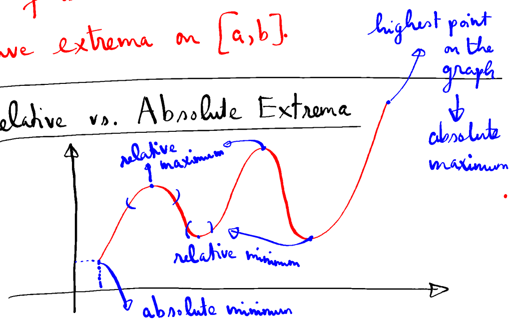
We say that $(c, f(c))$ is a maxima of f on $[a,b]$ if $f(x) \leq f(c)$ for every x in $[a,b]$

$\geq$

Extreme value theorem:

If f is continuous on $[a,b]$, then f will have extrema on $[a,b]$.

Relative vs. Absolute Extrema

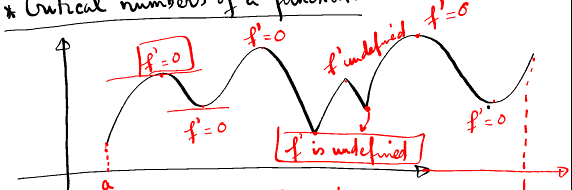


We say that a point $(c, f(c))$ is a relative maximum if there is an interval I around c such that $f(x) \leq f(c)$ for every x in I

$f(x) = x^3 - x$ on $[1,2]$

1.1 1.2 3 1.4

* Critical numbers of a function:



* Definition of a critical number of a function.

Suppose that $f(c)$ is defined.
We say that $x = c$ is a critical number of f if either $f'(c) = 0$ or $f'(c)$ is undefined.

E.g. $f(x) = x^4 - 8x^2$.
Find all the critical numbers of f?

$f'(x) = 4x^3 - 16x$

$f'(x) = 0$

$4x^3 - 16x = 0$

$4x(x^2 - 4) = 0$

$4x(x-2)(x+2) = 0$

$\boxed{x = 0}$; $\boxed{x = 2}$, $\boxed{x = -2}$

Ans: critical numbers are $x = 0$; $x = 2$; $x = -2$

$f(x) = \frac{4x}{x^2+1}$. Find critical #s of f

$f'(x) = \frac{4\cdot(x^2+1) - 4x\cdot(2x)}{(x^2+1)^2}$

$f'(x) = \frac{4x^2+4-8x^2}{(x^2+1)^2} = \boxed{\frac{-4x^2+4}{(x^2+1)^2}}$

f' undefined if $(x^2+1)^2 = 0$; $x^2+1=0$; $x^2=-1$.

$f'=0$ if $\frac{-4x^2+4}{(x^2+1)^2} = 0$; if $-4x^2+4=0$

$-4(x^2-1)=0$

$-4(x-1)(x+1)=0$

$\boxed{x=1}, \boxed{x=-1}$ ← critical numbers.

Theorem: Relative extrema of a function occur only at the critical #'s of that function.

Find ~~extrema~~ absolute extrema of a function f on $[a,b]$

1. Find all the critical numbers of f on $[a,b]$
2. Evaluate f at all those critical #s from part 1
3. Evaluate f at the endpoints a and b. That is, find $f(a)$ and $f(b)$
4. Compare the values from 2 and 3. The smallest value is the absolute minimum. The largest value is the absolute maximum.

$f(x) = 2x^3 - 6x$ on $\boxed{[0,3]}$

Find absolute maxima/minima of the function on the given interval.

1. Find critical #s: $f'(x) = 6x^2 - 6$

$f'(x) = 0$; $6x^2-6=0$

$6x^2 = 6$

$x^2 = 1$; $x = \pm 1$

$x=1$; $\cancel{x=-1}$

2. $f(1) = 2\cdot(1)^3 - 6\cdot 1 = \boxed{-4}$

$f(0) = 2\cdot 0^3 - 6\cdot 0 = 0$

$f(3) = 2\cdot(3)^3 - 6\cdot 3 = 2\cdot 27 - 18 = 54 - 18 = \boxed{36}$

* Conclusion:

Absolute minimum $= -4$.
It occurs where $x=1$

Absolute maximum $= 36$
It occurs where $x=3$

$f(x) = \sqrt[3]{x}$. on $[-8,8]$. Find absolute max/min of f on $[-8,8]$.

1. Critical #s: $f(x) = x^{1/3}$

$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$

f' is undefined when $x=0$

$x=0$ is a critical #

2. $f(0) = \sqrt[3]{0} = 0$

3. $f(-8) = \sqrt[3]{-8} = \boxed{-2}$; $f(8) = \sqrt[3]{8} = \boxed{2}$

abs. min abs. max

$f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$.

(1) $f(x) = (4 - x^2)^{1/2}$

$f'(x) = \frac{1}{2}(4 - x^2)^{-1/2} \cdot (-2x)$

$= \frac{1}{\cancel{2}\sqrt{4 - x^2}} \cdot (-\cancel{2}x)$

$\boxed{\boxed{f'(x)} = \frac{-x}{\sqrt{4 - x^2}}}$ $\boxed{2, -2}$ 0

$x = 0$; $x = 2$; $x = -2$

(2) $f(0) = \sqrt{4 - 0^2} = \sqrt{4} = 2$

$f(2) = \boxed{0}$

$f(-2) = \boxed{0}$

(3) ✓

(4) Abs. min $= 0$ it occurs when $x = 2$; $x = -2$

Abs. max $= 2$ it occurs when $x = 0$.