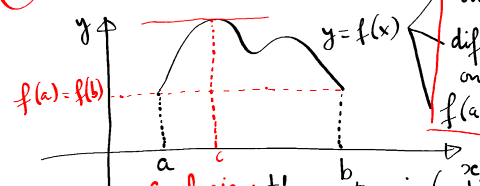
3.2. Rolle's Theorem and the Mean Value Theorem (MVT)

conditions

① Rolle's Theorem



- continuous on $[a,b]$
- differentiable on (a,b)
- $f(a)=f(b)$

Conclusion: there exists c in (a,b) s.t. $f'(c)=0$

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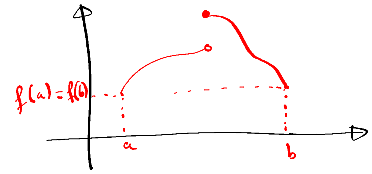
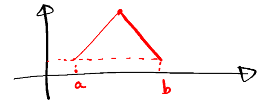
f is not continuous somewhere in (a,b)



f is not differentiable somewhere in (a,b)



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E.g. $f(x)=x^2-8x+5.$ on $[2,6]$.

Determine whether Rolle's theorem applies to this function. If it does, find a number c in $(2,6)$ such that $f'(c)=0$.

Does Rolle's theorem apply here? $\boxed{\text{Yes}}$

① Is f cont. on $[2,6]$? Yes

② Is f differentiable on $(2,6)$? Yes

③ Is $f(2)=f(6)$? Yes

$f(2)=-7. \quad f(6)=-7$

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$f'(x)=2x-8=0$

$2x=8$

$x=4.$

$\boxed{c=4}$; $f'(c)=0.$

E.g. $f(x)=\dfrac{x^2-1}{x}$ on $[-1,1]$

Does Rolle's Theorem apply? $\boxed{\text{NO}}$

f is not continuous at $x=0$ which is in $[-1,1]$

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E.g. $f(x)=\cos x.$ on $[0,2\pi]$

Does Rolle's Theorem apply? Yes

① cont. on $[0,2\pi]$

② diff. on $[0,2\pi]$

③ $f(0)=\cos(0)=1.$, $f(2\pi)=\cos(2\pi)=1$ $\}$ $f(0)=f(2\pi)$

Find all #s c such that $f'(c)=0$ in $(0,2\pi)$

$f'(x)=-\sin x=0$; $\sin x=0$

$x=\pi.$ $\boxed{c=\pi}$.

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E.g. $C(x)=10\left(\dfrac{1}{x}+\dfrac{x}{x+3}\right)$ on $[3,6]$

cost function of a production process.

x = # of units produced

① Does Rolle's Theorem apply to this function?

② If it does, then it says that the rate of change of cost $=0$ at some production level c in $(3,6)$. Find that production level.

$C(x)=10\cdot\left(\dfrac{1}{x}+\dfrac{x}{x+3}\right)$; $[3,6]$

f is not cont. at $x=0$; $x=-3$

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① f is cont. on $[3,6]$

② $f(x) = 10\cdot\left(\frac{1}{x} + \frac{x}{x+3}\right)$

$$f'(x) = 10\cdot\left(-\frac{1}{x^2} + \frac{1\cdot(x+3) - x\cdot 1}{(x+3)^2}\right)$$
$$= 10\cdot\left(-\frac{1}{x^2} + \frac{x+3-x}{(x+3)^2}\right)$$
$$= 10\cdot\left(-\frac{1}{x^2} + \frac{3}{(x+3)^2}\right)$$

f is not differentiable at $x=0$; $x=-3$

f is diff. on $(3,6)$

③ $f(3) = 10\cdot\left(\frac{1}{3}+\frac{1}{2}\right) = 10\cdot\left(\frac{5}{6}\right) = \frac{25}{3}$; $f(6) = 10\cdot\left(\frac{1}{6}+\frac{2}{3}\right) = 10\cdot\left(\frac{5}{6}\right) = \frac{25}{3}$.

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$f'(x) = 0$

$$10\cdot\left(-\frac{1}{x^2} + \frac{3}{(x+3)^2}\right) = 0$$
$$-\frac{1}{x^2} + \frac{3}{(x+3)^2} = 0$$
$$-\frac{1}{x^2} = \frac{-3}{(x+3)^2}$$
$$+(x+3)^2 = +3x^2$$
$$+(x^2+6x+9) = 3x^2$$
$$+x^2+6x+9 = 3x^2$$
$$2x^2 - 6x - 9 = 0$$
$$x = \frac{3}{2}(\sqrt{3}+1) \approx$$

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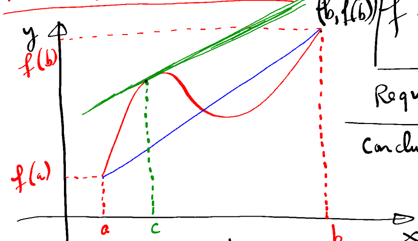
Mean Value Theorem



f cont. on $[a,b]$
diff. on (a,b)

Requirements

Conclusion?

There exists a number c in (a,b) s.t.

$$\underbrace{f'(c)}_{\text{slope of tangent line at } x=c} = \frac{f(b)-f(a)}{b-a}$$

→ slope of secant line connecting $(a,f(a))$ and $(b,f(b))$

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E.g. $f(x) = x^4 - 8x$ on $[0,2]$

MVT applies.

$f'(x) = 4x^3 - 8$

Want to find c such that:

$$f'(c) = \frac{f(2)-f(0)}{2-0}$$
$$4c^3 - 8 = \frac{(16-16) - (0 - 8\cdot 0)}{2} = 0$$

$4c^3 - 8 = 0$; $4c^3 = 8$; $c^3 = 2$

$\boxed{c = \sqrt[3]{2}}$

Q: $c^2 = 2$
$(\sqrt{2})^2 = 2$
$(-\sqrt{2})^2 = 2$
$(\sqrt[3]{2})^3 = 2$
$(-\sqrt[3]{2})^3 = -2$

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E.g. $f(x) = \cos x + \tan x$ on $[0,\pi]$

Does MVT apply here? NO

$\tan x = \frac{\sin x}{\cos x}$; $x = \frac{\pi}{2}$.

f is not cont. at $x = \frac{\pi}{2}$

E.g. Use the MVT to prove inequalities.

$|\cos a - \cos b| \le |a-b|$ for all real #s a,b.

Consider $f(x) = \cos x$ on $[a,b]$
cont. on $[a,b]$
diff. on (a,b)

MVT applies.

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There is a number c in (a,b) s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$f(x) = \cos x$

$f'(x) = -\sin x$

$$|-\sin c| = \left|\frac{\cos b - \cos a}{b-a}\right|$$
$$\Rightarrow \left|\frac{\cos b - \cos a}{b-a}\right| = |\sin c| \le 1$$
$$\left|\frac{\cos b - \cos a}{b-a}\right| \le 1$$
$$\boxed{|\cos b - \cos a| \le |b-a|}$$

Q.E.D.

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