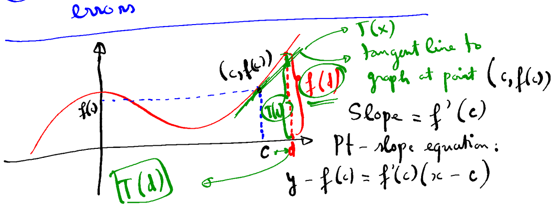
3.9 Differentials

1. Tangent line approximation.
2. The differential of a function.
3. Use the differential to estimate propagated errors



Slope $= f'(c)$

Pt-slope equation:

$y - f(c) = f'(c)(x - c)$

$T(d)$

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$y = \boxed{f'(c)(x-c) + f(c)}$ linear function of x

$T(x)$

$\boxed{\boxed{T(x)} = f'(c)(x-c) + f(c)}$

E.g. $f(x) = \sqrt{x}$. $c = 4$; $f(c) = \sqrt{4} = 2$

$(c, f(c)) = (4, 2)$.

$f'(x) = \frac{1}{2} x^{-1/2} = \boxed{\frac{1}{2\sqrt{x}}}$

$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

① Find $T(x)$ (the tangent line approximation) at $\boxed{(4,2)}$

$T(x) = \boxed{f'(4)}(x-4) + f(4)$

$\boxed{T(x) = \frac{1}{4}(x-4) + 2}$

$f(5) = \sqrt{5} = 2.236$

② $T(5) = \frac{1}{4}\cdot(5-4) + 2 = \frac{1}{4}\cdot 1 + 2 = \boxed{2.25}$

tangent line approx.

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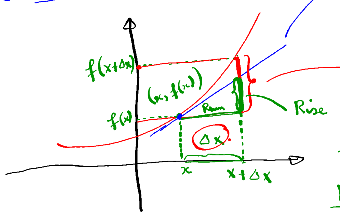
② The differential of a function

tangent line at $(x, f(x))$

$f'(x)$: slope.



$\boxed{\Delta y} = f(x+\Delta x) - f(x)$

$\frac{\text{Rise}}{\text{Run}} = f'(x)$

$\frac{\text{Rise}}{\Delta x} = f'(x)$

$\text{Rise} = f'(x)\Delta x$

Notation: $\boxed{dy = f'(x)\,\Delta x}$

dy gives a good approximation to Δy

dy is called the differential of the function $y = f(x)$

Notation: Set $\Delta x = dx$.

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$\boxed{dy = f'(x)\,dx}$

$dx = \Delta x$: change in x

$\boxed{dy = f'(x)\,dx}$: differential

$\Delta y = f(x+\Delta x) - f(x)$: change in y

$dy \approx \Delta y$.

E.g. $y = 6 - 2x^2$. $\quad dy := f'(x)\,dx$

① Calculate dy ?

$dy = f'(x)dx = \boxed{-4x\,dx}$

② When $x = -2$ and $\Delta x = dx = 0.1$.

Compare dy vs. Δy ?

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$dy = -4x\,dx$; $\boxed{x=-2}$ $\boxed{dx = 0.1}$

$dy = -4\cdot(-2)\cdot(0.1) = 8\cdot(0.1) = \boxed{0.8} = dy$

$\Delta y = f(x+\Delta x) - f(x)$; $\quad f(x) = 6 - 2x^2$

$= f(-2+0.1) - f(-2)$

$= f(-1.9) - f(-2)$

$= \underbrace{(6 - 2\cdot(-1.9)^2)}_{-1.22} - \underbrace{(6 - 2\cdot(-2)^2)}_{-2}$

$= -1.22 + 2 = \boxed{0.78} = \Delta y$

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E.g. Find differentials.

① $y = \csc(2x)$. $\quad dy \overset{?}{=} \boxed{f'(x)}\,dx$

$= \boxed{-\cot(2x)\csc(2x)\cdot 2\cdot dx}$

② $y = \boxed{\frac{\sec^2 x}{x^2+1}}$. $\quad dy \overset{?}{=} f'(x)\,dx$

$= \frac{(\sec^2 x)'\cdot(x^2+1) - (\sec^2 x)\cdot(2x)}{(x^2+1)^2}$

$= \frac{2\sec x\cdot\sec x\cdot\tan x\cdot(x^2+1) - 2x\cdot\sec^2 x}{(x^2+1)^2}$

$y = \sec^2 x$; $\boxed{u = \sec x}$ $\boxed{y = u^2}$

$\frac{dy}{dx}$

$y - u - x$

$= 2\sec x\cdot\sec x\tan x$

$2u\cdot\sec x\tan x$

$dy/dx = \frac{dy}{du}\frac{du}{dx}$

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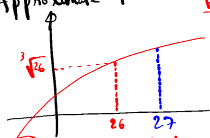
$$\boxed{dy = \frac{2\sec^2 x \tan x\,(x^2+1) - 2x\sec^2 x}{(x^2+1)^2}\,dx}$$

Use $\boxed{\text{differentials}}$ to approximate value of expression.

Approximate $\sqrt[3]{26}$



$\sqrt[3]{26} = 2.96296$

$\boxed{f(x) = \sqrt[3]{x}}$ $\rightarrow$ $f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3(\sqrt[3]{x})^2}$

$\boxed{x = 27}$. $\boxed{\Delta x \overset{?}{=} -1}$.

$f(27) = \sqrt[3]{27} = 3.$

$\Delta y = f(x+\Delta x) - f(x)$

$\Delta y = f(27-1) - f(27)$

$\Delta y = \boxed{f(26)} - 3$

approximation to

$dy = \boxed{f'(x)}dx = \frac{1}{27}\cdot(-1) = -\frac{1}{27}$

$f'(27) = \frac{1}{3\cdot(\sqrt[3]{27})^2} = \frac{1}{3\cdot 9} = \frac{1}{27}$

$-\frac{1}{27} \approx f(26) - 3$

$\boxed{f(26)} \approx 3 - \frac{1}{27} = \boxed{2.96296}$

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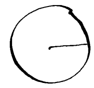
Using differential to approximate $\boxed{\text{error propagation}}$



Measure a radius of a circle.

To calculate the area of that circle.

Measured value of radius : $\underline{x}$

Error is Δx

Exact value of the radius : $\boxed{x + \Delta x}$

Use x to calculate area: $\underline{f(x) = \pi x^2}$ $\rightarrow$ measured area

Exact area : $f(x+\Delta x)$

$\Delta y = \boxed{\text{Propagated error}} = \underbrace{f(x+\Delta x)}_{\text{exact area}} - \underbrace{f(x)}_{\text{measured area.}}$

dy $\boxed{dy = f'(x)\,dx}$ $\longrightarrow$ approximation for propagated error.

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$$\boxed{\frac{\boxed{dy}}{\boxed{y}}}$$ percent of error

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