4.2. Area

Antiderivative ⟶ ans: family of functions

indefinite integral

(1) Sigma Notation.

$1+2+3+4+5+\cdots+1000 = \boxed{\sum_{i=1}^{1000} i} =$

$1^2+2^2+3^2+4^2+\cdots+9999^2 = \sum_{i=1}^{9999} i^2$

$\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\cdots+\frac{1}{26000^2} = \boxed{\sum_{i=2}^{26000} \boxed{\frac{1}{i^2}}}$

$\boxed{\frac{1}{\theta^2}+\frac{1}{\theta^2}+\frac{1}{\theta^2}+\frac{1}{\theta^2}+\cdots+\frac{1}{26000^2}}$

Aug 2-2:06 PM

$\boxed{\sum_{j=3}^{20}(j^2+1)} = (3^2+1)+(4^2+1)+(5^2+1)+\cdots + (20^2+1)$

$\sum_{i=1}^{10000}\left[(i-1)^2+(i+1)^2\right] = 2^2+\left[(2-1)^2+(2+1)^2\right]+\left[(3-1)^2+(3+1)^2\right] + \cdots + \left[(10-1)^2+(10+1)^2\right]$

(2) Summation formula

E.g. $\sum_{i=1}^{45} \boxed{2} = \underbrace{2+2+2+2+\cdots 2}_{45\text{ times}} = 2\cdot 45 = 90$

(1) If c is a constant, then $\boxed{\sum_{i=1}^{n} c = cn}$

Aug 2-2:17 PM

$\sum_{i=2}^{45} 2 = \underbrace{2+2+2+\cdots+2}_{44\text{ times}} = 2\cdot 44 = 48$

(2) $\boxed{\sum_{i=1}^{n} i}$

$1+2+3+\cdots+n$

Gauss $\quad 1+2+3+\cdots+100$

$\quad 100+99+98+\cdots+1 \quad +$

$\underbrace{101+101+101+\cdots+101}_{100} = 100\cdot 101$

$\frac{100\cdot 101}{2} = 50\cdot 101 = 5050$

Aug 2-2:23 PM

$\sum_{i=1}^{n} i = 1+2+3+\cdots+n = X \quad +$

$n+(n-1)+(n-2)+\cdots+1 = X$

$\underbrace{(n+1)+(n+1)+(n+1)+\cdots+(n+1)}_{n} = 2X$

$n\cdot(n+1) = 2X \Rightarrow X = \frac{n(n+1)}{2}$

(2) $\boxed{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}}$

(3) $\boxed{\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}}$

$\sum_{i=1}^{7} i^2 = \frac{7\cdot 8\cdot 15}{6} = 140$

Aug 2-2:27 PM

E.g. $\sum_{i=1}^{16}(5i-4) = \sum_{i=1}^{16} 5i - \boxed{\sum_{i=1}^{16} 4}$

$= 5\cdot\sum_{i=1}^{16} i - 4\cdot 16$

$= 5\cdot\frac{16\cdot(16+1)}{2} - 64$

$= 5\cdot 8\cdot 17 - 64 = 680 - 64$

$= 516$

E.g. $\sum_{i=1}^{10}\boxed{(i+1)^2} \qquad (i+1)(i+1) = i^2+2i+1$

$= \sum_{i=1}^{10}(i^2+2i+1) = \boxed{\sum_{i=1}^{10} i^2} + \sum_{i=1}^{10} 2i + \sum_{i=1}^{10} 1$

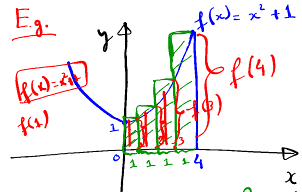
$= \boxed{\frac{10\cdot 11\cdot 21}{6}} + 2\cdot\frac{10\cdot 11}{2} + 10 = \cdots$

Aug 2-2:32 PM

(3) Area

E.g.

Find area under the graph of $f(x) = x^2+1$ on $[0, 4]$



* Approximation to this area.

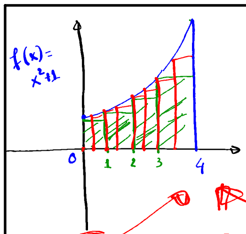
upper sum

Width of each rectangle is 1

$\boxed{1\cdot f(1) + 1\cdot f(2) + 1\cdot f(3) + 1\cdot f(4)} = \Sigma$ area of 4 rectangles

$2+5+10+17 = \boxed{34}$ ⟵ over estimation of the exact area.

Aug 2-2:39 PM



Lower Sum $= 1\cdot f(0) + 1\cdot f(1) + 1\cdot f(2) + 1\cdot f(3)$

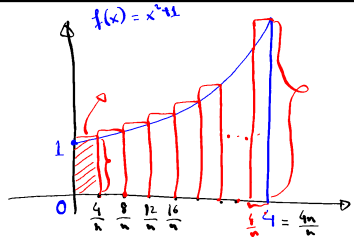
$= 1 + 2 + 5 + 10$

$= \boxed{18}$ ← under estimation of area.

$18 < \boxed{\text{area}} < 34$

Limit of sum of the areas of these small rectangles as the number of rectangles approaches infinity

Aug 2-2:46 PM



Divide the interval $[0,4]$ into n. subintervals

width of a rectangle?

width $= \frac{4}{n}$.

1st rectangle area $= \frac{4}{n}\cdot f\left(\frac{4}{n}\right)$

$$\frac{4}{n}\cdot f\left(\frac{4}{n}\right) + \frac{4}{n}\cdot f\left(\frac{8}{n}\right) + \frac{4}{n}\cdot f\left(\frac{12}{n}\right) + \frac{4}{n}\cdot f\left(\frac{16}{n}\right) + \dots + \frac{4}{n}\, f\left(\frac{4n}{n}\right)$$

$4 \to 8 \to 12 \to 16 \to \dots 4n$

$\boxed{\sum_{i=1}^{n} \frac{4}{n} f\left(\frac{4i}{n}\right)}$ ← upper sum for n rectangles

Aug 2-2:52 PM

Precise Area under the curve $= \boxed{\lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n} f\left(\frac{4i}{n}\right)}$ $f(x) = x^2+1$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n}\cdot\left[\left(\frac{4i}{n}\right)^2 + 1\right]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n}\left[\frac{16i^2}{n^2} + 1\right]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n}\left(\frac{64i^2}{n^3} + \frac{4}{n}\right) = \lim_{n\to\infty}\left(\boxed{\sum_{i=1}^{n} \frac{64 i^2}{n^3}} + \sum_{i=1}^{n}\frac{4}{n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{64}{n^3}\cdot\boxed{\sum_{i=1}^{n} i^2} + \frac{1}{n}\cdot\sum_{i=1}^{n} 4\right)$$

$$= \lim_{n\to\infty}\left(\boxed{\frac{64}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}} + \frac{1}{\cancel{n}}\cdot 4\cancel{n}\right)$$

$$\frac{128}{6} + 4 = \frac{64}{3} + 4 = \boxed{\frac{76}{3}}$$

Aug 2-3:01 PM

Summary of find area under $f(x)$ on $[a,b]$ using limit.

1. Divide $[a,b]$ into n subintervals. width of each subinterval $\frac{b-a}{n}$.
2. Expression for the sum of the areas of n rectangles. $\sum_{i=1}^{n} \frac{b-a}{n}\cdot f\left(a + \frac{b-a}{n}\cdot i\right)$
3. $\lim_{n\to\infty}$ of the sum. → area.

Aug 2-3:09 PM