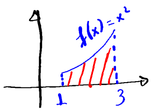
4.4 Fundamental Theorems of Calculus

1 F.T.C. I

$\int_1^3 \boxed{x^2}\, dx$ — integrand

find the antiderivative of the integrand. $\frac{x^3}{3}$.

Evaluate the antiderivative at the upper and lower bounds and find the difference.



$$\frac{3^3}{3} - \frac{1^3}{3} = \frac{27}{3} - \frac{1}{3} = \boxed{\frac{26}{3}}$$

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$$\int_0^1 \boxed{\sqrt{x}}\, dx = \int_0^1 \boxed{x^{\frac12}}\, dx = \frac{x^{\frac12+1}}{\frac12+1}\Bigg|_0^1$$

$$= \frac{x^{\frac32}}{\frac32}\Bigg|_0^1 = \frac23 x^{\frac32}\Bigg|_0^1$$

$$= \frac23\cdot(1)^{\frac32} - \frac23\cdot(0)^{\frac32}$$

$$= \frac23 - 0 = \boxed{\frac23}$$

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F.T.C. I.

If f is a continuous function on $[a,b]$ and $\boxed{F}$ is an antiderivative of f on $[a,b]$ $(F' = f)$

then $\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$

E.g. $\int_1^2 (6x^2 - 3x)\,dx = \left(6\int_1^2 x^2 - 3\int_1^2 x\right)dx = 6\cdot\frac{x^3}{3}\Big|_1^2 - 3\cdot\frac{x^2}{2}\Big|_1^2$

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$$6\cdot\left(\frac{2^3}{3} - \frac{1^3}{3}\right) - 3\cdot\left(\frac{2^2}{2} - \frac{1^2}{2}\right) = 6\cdot\frac73 - 3\cdot\frac32 = \frac{21}{3} - \frac92$$

$$= \frac{42-27}{6} = \boxed{\frac{25}{6}}$$

E.g. $\int_{-2}^{-1}\left(u - \frac{1}{u^2}\right)du = \int_{-2}^{-1}\left(\boxed{u - u^{-2}}\right)du$

$$= \left(\frac{u^2}{2} - \frac{u^{-1}}{-1}\right)\Bigg|_{-2}^{-1}$$

$$= \left(\frac{u^2}{2} + \frac1u\right)\Bigg|_{-2}^{-1} = \frac{(-1)^2}{2} + \frac{1}{-1} - \left(\frac{(-2)^2}{2} + \frac{1}{-2}\right)$$

$$= \frac12 - 1 - 2 + \frac12 = \boxed{-2}.$$

E.g. $\int_0^{\pi/4} \frac{\boxed{1 - \sin^2\theta}^{\cos^2\theta}}{\cos^2\theta}\,d\theta = \int_0^{\pi/4}\frac{\cos^2\theta}{\cos^2\theta}\,d\theta = \int_0^{\pi/4} 1\,d\theta = \theta\Big|_0^{\pi/4}$

$$= \boxed{\frac{\pi}{4}}.$$

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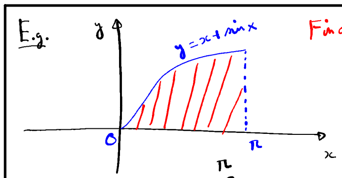
E.g. Find the area?



$$\text{Area} = \int_0^\pi \left(\boxed{x + \sin x}\right)dx = \left(\frac{x^2}{2} - \cos x\right)\Bigg|_0^\pi$$

$$= \left(\frac{\pi^2}{2} - \cos\pi\right) - \left(\frac{0^2}{2} - \cos 0\right)$$

$$= \frac{\pi^2}{2} + 1 + 1 = \boxed{\frac{\pi^2}{2} + 2}$$

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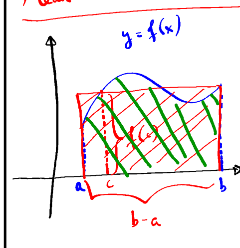
Mean Value Theorem for integrals



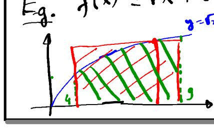
f: continuous on $[a,b]$

$\int_a^b f(x)\,dx$ = area under f on $[a,b]$

MVT for integrals says that there exists a number c in $[a,b]$ such that $\boxed{\int_a^b f(x)\,dx} = \boxed{f(c)\cdot(b-a)}$

area of rectangle height $f(c)$, width $(b-a)$

E.g. $f(x) = \sqrt{x}$ on $[4,9]$.



$$\int_4^9 \sqrt{x}\,dx = \int_4^9 \boxed{x^{1/2}}\,dx = \frac23 x^{3/2}\Bigg|_4^9$$

$$= \frac23\left((\sqrt9)^3 - (\sqrt4)^3\right) = \boxed{14}$$

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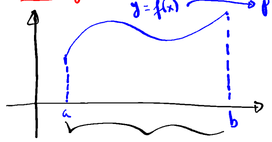
there is c in $[4,9]$ s.t.

$$f(c)\cdot[9-4] = 14$$

$$\sqrt{c}\cdot 5 = 14 \qquad \sqrt{c} = \frac{14}{5} \qquad c = \left(\frac{14}{5}\right)^2 = \boxed{7.84}$$

Average Value of a function



$y=f(x) \to$ profit function

$$\frac{\int_a^b f(x)\,dx}{b-a}$$

$\int_a^b f(x)\,dx \to$ total profit over the period of time from a to b

$b-a \to$ length of time interval

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Average value of f on $[a,b]$ = $\boxed{\dfrac{\int_a^b f(x)\,dx}{b-a}}$

E.g. $f(x) = \cos x$ on $[0, \frac{\pi}{2}]$

Find average value of f on this interval.

$$= \frac{\int_0^{\pi/2} \cos x\,dx}{\frac{\pi}{2}} = \frac{\sin x\Big|_0^{\pi/2}}{\frac{\pi}{2}} = \frac{\sin\frac{\pi}{2} - \sin 0}{\frac{\pi}{2}}$$

$$= \frac{1}{\frac{\pi}{2}} = \boxed{\frac{2}{\pi}}$$

F.T.C. II

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$$\int_0^{x} (t^3 + 2t - 2)\,dt = \left(\frac{t^4}{4} + t^2 - 2t\right)\Big|_0^x$$

$$F(x) = \frac{x^4}{4} + x^2 - 2x$$

$$F'(x) = \boxed{x^3 + 2x - 2}$$

F.T.C. II.

$$\frac{d}{dx}\left(\underbrace{\int_a^x f(t)\,dt}_{\text{a function of } x}\right) = f(x)$$

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E.g. $\dfrac{d}{dx}\left(\displaystyle\int_{-1}^{x} \frac{t^2}{t^2+1}\,dt\right) \overset{?}{=} \dfrac{x^2}{x^2+1}$

Differentiation and Integration are inverse operations.

$$\frac{d}{dx}\left(\int_0^{x} \sec^3 t\,dt\right) \overset{?}{=} \sec^3 x.$$

$$\frac{d}{dx}\left(\int_1^{x^2} t\sin(t)\,dt\right) = \left(x^2\sin(x^2)\right)\cdot 2x$$

Reason why we have to multiply by the derivative of x^2

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$$\frac{d}{dx}\Big(\underbrace{\int_1^{x^4} t\sin(t)\,dt}_{F(x)}\Big) \to \boxed{u = x^2}$$

F — u — x

$\dfrac{dF}{dx}$ F.T.C.2 $\dfrac{dF}{dx} = \dfrac{dF}{du}\cdot\dfrac{du}{dx}$ (Chain Rule)

$$\frac{dF}{dx} = \boxed{\frac{d}{du}\left(\int_1^{u} t\sin(t)\,dt\right)}\cdot 2x$$

$$= u\sin(u)\cdot 2x = \boxed{x^2\sin(x^2)\cdot 2x}$$

E.g. $\dfrac{d}{dx}\left(\displaystyle\int_0^{\cos x} \sin(t^2)\,dt\right) = \sin(\cos^2 x)\cdot(-\sin x)$

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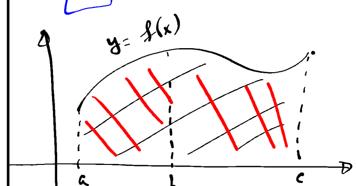
E.g. $\dfrac{d}{dx}\left(\displaystyle\int_{-x}^{x^3} t^3\,dt\right) = \dfrac{d}{dx}\left(\displaystyle\int_{-x}^{0} t^3\,dt + \int_0^{x^3} t^3\,dt\right)$



$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

$$\frac{d}{dx}\left(\int_{-x}^{0} t^3\,dt\right) + \frac{d}{dx}\left(\int_0^{x^3} t^3\,dt\right)$$

$$\frac{d}{dx}\left(-\int_0^{-x} t^3\,dt\right) \qquad -(-x)^3\cdot(-1) = -x^3$$

$$-x^3 + x^6\cdot 2x = \boxed{-x^3 + 2x^7}$$

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