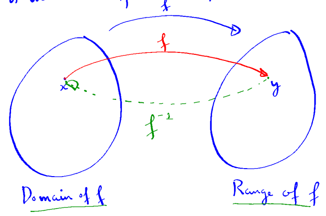
5.3 - Inverse Functions

What is the inverse function of a function f



f and g are inverse functions if

$$f(g(x)) = x \text{ and } g(f(x)) = x$$

E.g. $f(x) = \sin x$ $\quad \sin(\frac{\pi}{2}) = 1 \quad ; \quad \sin(\frac{\pi}{6}) = \frac{1}{2}$

$g(x) = \arcsin x$ $\quad \arcsin(1) = \frac{\pi}{2} \quad \arcsin(\frac{1}{2}) = \frac{\pi}{6}$.

Aug 10-12:02 PM

$\arcsin(\sin x) = x \quad \sin(\arcsin x) = x$.

E.g. $f(x) = \ln x$ $\quad \ln 1 = 0 \quad \ln(e) = 1$

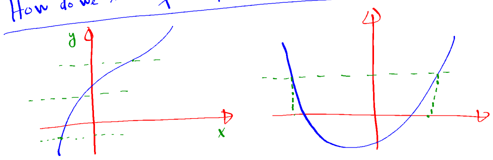
$g(x) = e^x$. $\quad e^0 = 1 \quad e^1 = e$.

$\boxed{\ln(e^x) = x} \quad \boxed{e^{\ln x} = x}$

E.g. $f(x) = \sqrt[3]{x}$; $\sqrt[3]{27} = 3$

$g(x) = x^3$ $\quad 3^3 = 27$.

How do we know if a function has an inverse?



Aug 10-12:07 PM

f has an inverse on an interval if and only if f is one to one

If f is strictly monotonic (f is strictly increasing / f is strictly decreasing), then f is one-to-one. Hence, it will have an inverse.

E.g. $f(x) = \sqrt{2x-3}$ on $[\frac{3}{2}, \infty)$

$$f'(x) = \frac{\cancel{2}}{\cancel{2}\sqrt{2x-3}} = \frac{1}{\sqrt{2x-3}} \text{ on } (\tfrac{3}{2}, \infty);$$

$f'(x) > 0$

Since $f' > 0$ on $(\frac{3}{2}, \infty)$, f is strictly increasing

This implies that f has an inverse.

Aug 10-12:11 PM

$f(x) = \sqrt{2x-3}$. $\quad f^{-1}(x) \stackrel{?}{=}$

$y = \sqrt{2x-3}$

Solve for x in terms of y.

$y^2 = 2x - 3$; $\quad x = \frac{y^2+3}{2}$; interchange x and y.

$$f^{-1}(x) = \frac{x^2+3}{2}$$

Inverse function Theorem

If f is differentiable on an interval I and f has an inverse function g on I, then

$$\boxed{g'(x) = \frac{1}{f'(g(x))}} \text{ provided } f'(g(x)) \neq 0$$

Aug 10-12:15 PM

If we use the notation f^{-1} for the inverse function, then the formula becomes,

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Reason why formula is true:

$$g'(x) = \frac{1}{f'(g(x))}$$

We know g and f are inverses.

$$f(g(x)) = x$$

Take the derivative with respect to x of both sides.

$\underbrace{f'(g(x)) \cdot \boxed{g'(x)}}_{\text{Chain Rule}} = 1$. This implies: $\boxed{g'(x) = \frac{1}{f'(g(x))}}$.

Aug 10-12:19 PM

E.g. $f(x) = x^3 + 2x - 1$. $\quad$ Want $g(2) = ?$

(1) Verify that f has an inverse g. $\quad f(1) = 2$

(2) Find $g'(2)$? $\quad g(2) = 1$

* $\boxed{f'(x) = 3x^2 + 2} > 0$

Hence, f is increasing. Hence, f is one-to-one.

Hence, f has an inverse.

* $g'(2) = \frac{1}{f'(g(2))}$ $\quad$ (By the inverse function theorem)

$g(2) \stackrel{?}{=} 1$ $\quad g'(2) = \frac{1}{f'(1)} = \frac{1}{5}$.

Aug 10-12:23 PM

Ex ① $f(x)=\sqrt{x-4}$ Find $(f^{-1})'(2) = ?\ 4$

② $f(x)=\frac{x+3}{x+1}$; $x>-1$ $(f^{-1})'(2) \overset{?}{=} \boxed{-2}$ $f(1)=2$ $f^{-1}(2)=1$

Using the inverse function theorem.

$$(f^{-1})'(2) = \frac{1}{f'(\boxed{f^{-1}(2)})} = \frac{1}{f'(8)} = \frac{1}{\frac{1}{2\sqrt{8-4}}} = \frac{1}{\frac{1}{4}} = 4$$

$f(x)=\sqrt{x-4}$; $f'(x)=\frac{1}{2\sqrt{x-4}}$.

$f^{-1}(2)=?$. Find a # ? such that $f(?)=2$

$?=8$.

that is, $f(8)=2$. Hence, $f^{-1}(2)=8$

② $(f^{-1})'(2)=\frac{1}{f'(\boxed{f^{-1}(2)})}$; $f'(x)=\frac{x+1-(x+3)}{(x+1)^2}=\frac{-2}{(x+1)^2}$

Aug 10-12:30 PM

$$(f^{-1})'(2) = \frac{1}{f'(1)} = \frac{1}{\frac{-2}{(1+1)^2}} = \frac{1}{\frac{-2}{4}} = \frac{1}{-\frac{1}{2}} = -2$$

Derive the formula for the derivative of e^x, arcsin x.

* Derivative of e^x

$f(x)=\ln x$ $g(x)=e^x$. We want : $g'(x)=?$

By the inverse function theorem: $g'(x)=\frac{1}{f'(g(x))}$

$f'(x)=\frac{1}{x}$; $f'(g(x))=\frac{1}{e^x}$

$g'(x)=\frac{1}{\frac{1}{e^x}}=e^x$

The derivative of the function e^x is e^x.

Aug 10-12:35 PM

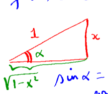
$$\boxed{\frac{d}{dx}(e^x)=e^x \quad ; \quad \frac{d}{dx}(e^u)=e^u\cdot\frac{du}{dx}}$$

Formula for the derivative of arcsin x

$f(x)=\sin x$; $g(x)=\arcsin x$. Want: $g'(x)=?$

By I.F.T $g'(x)=\frac{1}{f'(g(x))}=\frac{1}{\cos(\arcsin x)}=\frac{1}{\sqrt{1-x^2}}$

$f'(x)=\cos x$. $f'(g(x))=\boxed{\cos(\arcsin x)}$



$\sin\alpha = x$

$\arcsin x=\alpha$

$\cos\alpha = ?\ \sqrt{1-x^2}$

$\cos(\arcsin x)=\sqrt{1-x^2}$

$$\boxed{\frac{d}{dx}(\arcsin x)=\frac{1}{\sqrt{1-x^2}}}$$

Aug 10-12:43 PM