5.4 Differentiation and Integration of Exp. function base e.

Differentiation

$$\boxed{\frac{d}{dx}(e^x) = e^x \qquad (e^x)' = e^x \qquad \frac{d}{dx}(e^u) = e^u \cdot \frac{du}{dx} \qquad (e^u)' = e^u \cdot u'}$$

E.g. $y = e^{\boxed{-x}}$. $y' = e^{-x}\cdot(-1) = -e^{-x}$

$y = e^{-2x^3}$. $y' = -6x^2 e^{-2x^3}$.

$y = \boxed{x^2}\,\boxed{e^{-x}}$ $\quad y' = 2x\cdot e^{-x} + (-e^{-x})\cdot x^2$

$= 2xe^{-x} - x^2e^{-x} = xe^{-x}(2-x)$

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$y = \ln\left(\frac{1+e^x}{1-e^x}\right)$. Find $y' = ?$

$y = \boxed{\ln(1+e^x)} - \ln(1-e^x)$

$y' = \frac{(1+e^x)'}{1+e^x} - \frac{(1-e^x)'}{1-e^x} = \frac{e^x}{1+e^x} - \frac{-e^x}{1-e^x}$

$= \frac{e^x}{1+e^x} + \frac{e^x}{1-e^x} = \frac{e^x - \cancel{e^{2x}} + e^x + \cancel{e^{2x}}}{(1+e^x)(1-e^x)}$

$$\boxed{y' = \frac{2e^x}{(1+e^x)(1-e^x)}}$$

$\frac{d}{dx}\left(\int_0^{e^{2x}} \ln(1+t)\,dt\right) \underset{\text{F.T.C.2}}{=} \left(\ln(1+e^{2x})\right)\cdot\left(2e^{2x}\right)$

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$\frac{d}{dx}\left(\int_{e^{x/2}}^{e^{2x}} \ln(1+t)\,dt\right)$ $\qquad \int_{e^{x^2}}^{x}$

$\frac{d}{dx}\left(\int_{e^{x/2}}^{10} \ln(1+t)\,dt + \int_{10}^{e^{2x}} \ln(1+t)\,dt\right)$

$\frac{d}{dx}\left(-\int_0^{e^{x/2}} \ln(1+t)\,dt + \int_0^{e^{2x}} \ln(1+t)\,dt\right)$

$-\ln(1+e^{x/2})\cdot\frac{1}{2}e^{x/2} + \ln(1+e^{2x})\cdot 2e^{2x}$.

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E.g. Find the equation of the tangent line to the graph of $y = xe^x - e^x$ at the point $(1,0)$.

* $y' = e^x + xe^x - e^x$

$y'(1) = e + e - e = e$ ← Slope of tangent line at $(1,0)$

$y - 0 = e\cdot(x-1)$; $\boxed{y = ex - e}$.

E.g. Implicit Differentiation

$e^{xy} + x^2 - y^2 = 10$.

$\frac{dy}{dx} = ?$

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$e^{xy} + x^2 - y^2 = 10$

$\boxed{\frac{d}{dx}(e^{xy})} + 2x - 2y\frac{dy}{dx} = 0$

$e^{xy}\cdot\frac{d}{dx}(xy) + 2x - 2y\frac{dy}{dx} = 0$

$e^{xy}\cdot\left(y + x\frac{dy}{dx}\right) + 2x - 2y\frac{dy}{dx} = 0$

$\boxed{ye^{xy}} + \boxed{xe^{xy}\frac{dy}{dx}} + \boxed{2x}\boxed{-2y\frac{dy}{dx}} = 0$

$\left(xe^{xy} - 2y\right)\frac{dy}{dx} = -ye^{xy} - 2x$

$$\boxed{\frac{dy}{dx} = \frac{-ye^{xy} - 2x}{xe^{xy} - 2y}}$$

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Integration of exponential functions with base e

$\int e^x\,dx = e^x + C$; $\quad \int e^u\,du = e^u + C$

u: function of x.

E.g. $-\frac{1}{3}\int e^{\boxed{1-3x}}\underbrace{(-3dx)}_{du}$ $\quad$ let $u = 1-3x$; $du = -3dx$

$-\frac{1}{3}\int e^u\,du = -\frac{1}{3}e^u + C = \boxed{-\frac{1}{3}e^{1-3x} + C}$

$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$ $\quad$ let $u = e^x + e^{-x}$. $du = (e^x - e^{-x})\,dx$

$\int \frac{du}{u} = \ln|u| + C = \ln(e^x + e^{-x}) + C$.

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Ex. (1) $\int e^x (e^x+1)^2 \, dx$ (2) $\int e^{2x} \csc(e^{2x}) \, dx$

(3) $\int_0^{\sqrt{2}} x \cdot e^{-x^2/2} \, dx.$

Solution: (1) $\int e^x \boxed{\boxed{(e^x+1)}}^2 \, dx$ let $u = e^x + 1$, $du = e^x \, dx$

$\int u^2 \, du = \frac{u^3}{3} + C = \frac{(e^x+1)^3}{3} + C.$

(2) $\int \boxed{e^{2x}} \csc(\boxed{e^{2x}}) \boxed{dx}$ let $u = e^{2x}$. $du = 2e^{2x} \, dx$

$\frac{1}{2} \int \csc(e^{2x}) \underbrace{\boxed{2 \cdot e^{2x} \, dx}}_{du} = \frac{1}{2} \int \csc(u) \, du$

$= \frac{1}{2}\left(-\ln\left|\csc(u) + \cot(u)\right|\right) + C$

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$-\frac{1}{2} \ln\left|\csc(e^{2x}) + \cot(e^{2x})\right| + C.$

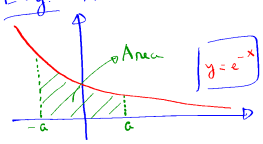
(3) $-\int_0^{\sqrt{2}} (-x) \, e^{-x^2/2} \, (dx)$ let $u = -\frac{x^2}{2}$. $du = -x \, dx$

$x = 0 \to \boxed{u = 0}$

$x = \sqrt{2} \to u = -1$

$-\int_0^{-1} e^u \, du = \int_{-1}^{0} e^u \, du = e^u \Big|_{-1}^{0} = 1 - e^{-1} = \boxed{1 - \frac{1}{e}}.$

E.g. Area



Find the # a such that the area in the picture is equal to $\boxed{\frac{8}{3}}$

$\int_{-a}^{a} e^{-x} \, dx = \frac{8}{3}.$

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$\int_{-a}^{a} e^{-x} \, dx = -e^{-x} \Big|_{-a}^{a} = -e^{-a} - (-e^{a})$

$= -e^{-a} + e^{a} = \frac{8}{3}.$

Solve the equation: $e^a - \boxed{e^{-a}} = \frac{8}{3}$. For a.

$e^a - \frac{1}{e^a} = \frac{8}{3}$

Multiply both sides by e^a:

$e^{2a} - 1 = \frac{8}{3} e^a$

$3e^{2a} - 3 = 8e^a$

$3e^{2a} - 8e^a - 3 = 0$

$3(e^a)^2 \boxed{-8} e^a - 3 = 0$

$(3e^a + 1)(e^a - 3) = 0$

Either $\cancel{\boxed{3e^a + 1 = 0}}$

or $e^a - 3 = 0$

$e^a = 3$

$\boxed{a = \ln 3}$

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