Section 5.5 Differentiation and Integration of exponential functions with base other than e.

(1) Differentiation

$$\boxed{\frac{d}{dx}\left(e^x\right) = e^x}$$

$y = 2^x$; $y = \pi^x$; $y = \left(\frac{1}{3}\right)^x$.

Derivative of $y = a^x$; $a > 0$.

$$\boxed{\frac{d}{dx}\left(a^x\right) = a^x \ln a}$$

Note: when $a = e$

$\frac{d}{dx}(e^x) = e^x \ln e = e^x$

$\frac{d}{dx}(2^x) = 2^x \ln 2$

$\frac{d}{dx}(\pi^x) = \pi^x \ln \pi$.

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Why the formula is true?

$y = a^x = \left(e^{\ln a}\right)^x = \boxed{e^{x \ln a}}$

$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\boxed{x \ln a}}\right) = e^{x \ln a} \cdot \left(x \ln a\right)' = e^{x \ln a} \cdot \ln a$

$= a^x \cdot \ln a$

$$\boxed{\frac{d}{dx}\left(a^x\right) = a^x \ln a \qquad \frac{d}{dx}\left(a^u\right) = a^u \ln a \cdot \frac{du}{dx} \quad u\text{: is a function of } x}$$

E.g. $y = 6^{3x-4}$. $y' = 6^{3x-4} \cdot (\ln 6) \cdot 3$

$y = \boxed{5^{-t}} \cdot \boxed{\sin(2t)}$ $\frac{dy}{dt} = 5^{-t} \cdot \ln 5 \cdot (-1) \cdot \sin(2t) + 5^{-t} \cdot 2\cos(2t)$

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$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$ $y = \log_2 x$; $\log_\pi x$; $\log_{\frac{1}{2}} x$.

$$\boxed{\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x \ln a}} \quad a > 0$$

Note: $\frac{d}{dx}(\ln x) = \frac{d}{dx}\left(\log_e x\right) = \frac{1}{x \ln e} = \frac{1}{x}$.

$\frac{d}{dx}\left(\log_2 x\right) = \frac{1}{x \ln 2}$; $\frac{d}{dx}\left(\log_\pi x\right) = \frac{1}{x \ln \pi}$

Note: $\log x := \log_{10} x$. $\frac{d}{dx}\left(\log x\right) = \frac{1}{x \ln 10}$.

Why is this formula true?

Inverse function theorem: $g(x)$ is the inverse function of $f(x)$

$$\boxed{g'(x) = \frac{1}{f'(g(x))}}$$

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$f(x) = a^x$; $g(x) = \boxed{\log_a x}$; $f'(x) = a^x \ln a$

I.F.T. $g'(x) = \frac{1}{f'(g(x))} =$

$= \frac{1}{a^{\log_a x} \ln a} = \frac{1}{x \ln a}$.

$$\boxed{\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x \ln a} \ ; \quad \frac{d}{dx}\left(\log_a u\right) = \frac{1}{u \ln a} \cdot \frac{du}{dx}}$$

E.g. $y = \log_2 \left(t^2+7\right)^3$. $\frac{dy}{dt} = 3 \cdot \frac{1}{(t^2+7) \ln 2} \cdot 2t$

$y = 3 \log_2 \boxed{(t^2+7)}$ $= \frac{6t}{(t^2+7) \ln 2}$.

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$y = \log_5 \frac{4}{x^2 \sqrt{1-x}}$; $y' = ?$

$y = \log_5 4 - \log_5 \boxed{x^2 \sqrt{1-x}} = \log_5 4 - \log_5 x^2 - \log_5 (1-x)^{1/2}$

$y = \log_5 4 - 2 \log_5 x - \frac{1}{2} \log_5 \boxed{(1-x)}$

$y' = -2 \cdot \frac{1}{x \ln 5} - \frac{1}{2} \cdot \frac{1}{(1-x) \ln 5} \cdot (-1)$

$= \frac{-2}{x \ln 5} + \frac{1}{2(1-x) \ln 5}$.

$$\left.\boxed{\begin{array}{ll} (a^x)' = a^x \ln a & (a^u)' = a^u \cdot \ln a \cdot u' \\ (\log_a x)' = \frac{1}{x \ln a} & (\log_a u)' = \frac{u'}{u \ln a} \end{array}}\right\} \begin{array}{l} u \text{ is} \\ \text{a function} \\ \text{of } x \end{array}$$

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Logarithmic Differentiation

$\boxed{y = x^x}$ y'

$(x^n)' = n x^{n-1}$: can't use Power rule b/c the exponent is not a constant

$(a^x)' = a^x \ln a$; can't use Exp. rule b/c the base is not a constant.

Take ln of both sides:

$\ln y = \ln x^x$. So, $\ln y = \boxed{x \ln x.}$

Take the derivative of both sides with respect to x:

$\frac{y'}{y} = \ln x + x \cdot \frac{1}{x}$. So, $\frac{y'}{y} = \ln x + 1$

Multiply both sides by y: $y' = (\ln x + 1) y = (\ln x + 1) x^x$.

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$$\boxed{(x^x)' = (\ln x + 1)\,x^x}$$

Ex. ① $y = (1+x)^{\frac{1}{x}}$ $y' = ?$

② $y = (\ln x)^{\cos x}$. Find the equation of the tangent line to the graph of y at $(e, 1)$.

Solution: ① Answer: $y' = \left(\frac{-\ln(1+x)}{x^2} + \frac{1}{x+x^2}\right)(1+x)^{1/x}$

$y' = \left(\left(x - \ln(1+x)(1+x)\right)\right)\cdot(1+x)^{1/x}$

$y' = (1+x)^{1/x}\,\dfrac{\frac{x}{1+x} - \ln(1+x)}{x^2}$

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$y = (1+x)^{\frac{1}{x}}$ $\frac{y'}{y} = -\frac{1}{x^2}\cdot\ln(1+x) + \frac{1}{x}\cdot\frac{1}{1+x}$

$\ln y = \frac{1}{x}\cdot\ln(1+x)$

$\frac{y'}{y} = \frac{-\ln(1+x)}{x^2} + \frac{1}{x(1+x)}$ $f(x)^{g(x)}$

$$\boxed{y' = \left(\frac{-\ln(1+x)}{x^2} + \frac{1}{x(x+1)}\right)(1+x)^{\frac{1}{x}}}$$

② $y = (\ln x)^{\cos x}$. $y' = \left(-\sin x\cdot\ln(\ln x) + \frac{\cos x}{x\ln x}\right)\cdot(\ln x)^{\cos x}$

$\ln y = \cos x\,\ln(\ln x)$ $\frac{y'}{y} = -\sin x\ln(\ln x) + \cos x\cdot\frac{1}{\ln x}\cdot\frac{1}{x}$

$y' = \left(-\sin x\ln(\ln x) + \frac{\cos x}{x\ln x}\right)(\ln x)^{\cos x}$

Evaluate y' at $x = e$. $\left(-\sin(e)\ln(\ln e) + \frac{\cos e}{e\ln e}\right)(\ln e)^{\cos e}$ $= \frac{\cos e}{e}$.

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$y - 1 = \frac{\cos e}{e}(x - e)$

$$\boxed{y = \frac{\cos(e)}{e}\cdot x - \cos(e) + 1.}$$

$\int e^x\,dx = e^x + C$ $(a^x)' = a^x\ln a$

$\int a^x\,dx = \frac{a^x}{\ln a} + C$

$\left(\frac{a^x}{\ln a}\right)' = \frac{1}{\ln a}(a^x)' = \frac{1}{\ln a}\cdot a^x\ln a = a^x$

$$\boxed{\int a^x\,dx = \frac{a^x}{\ln a} + C. \quad \int a^u\,du = \frac{a^u}{\ln a} + C}$$

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$-\int 8^{-x}(-dx)$ let $u = -x$. $du = -dx$

$-\int 8^u\,du = -\frac{8^u}{\ln 8} + C$.

$\int 2^{\sin x}\cos x\,dx$ let $u = \sin x$. $du = \cos x\,dx$

$\int 2^u\,du = \frac{2^u}{\ln 2} + C = \frac{2^{\sin x}}{\ln 2} + C$

Compound Interests

P: principal. r: interest rate.

t: # of years you put P in a saving account. A: balance after t years.

n: # of times P is compounded per year.

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$$\boxed{A = P\cdot\left(1 + \frac{r}{n}\right)^{nt}}$$

$\lim_{n\to\infty} P\cdot\left(1 + \frac{r}{n}\right)^{nt}$

$P\left[\lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^n\right]^t$

e^r

$$\boxed{\lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^n = e^r}$$

* Continuously compounded

$A = P\cdot e^{rt}$

E.g. Put P dollars in an investment account. $r = 5\%$. Compounded yearly. $n = 1$. $t = 30$ years. Want to have \$100,000 after 30 years. Find P?

$r = 0.05$. $n = 1$; $t = 30$; $A = 100{,}000$. $P = ?$

$100{,}000 = P\cdot\left(1 + \frac{0.05}{1}\right)^{30}$

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$100000 = P\cdot(1.05)^{30}$

$P = \frac{100\,000}{(1.05)^{30}} = 23137.74$

* Compounded continuously.

$A = P\cdot e^{rt}$

$100{,}000 = P\cdot e^{(0.05)30}$

$P = \frac{100{,}000}{e^{(0.05)\cdot 30}} \approx 22313.$

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