5.6. Integrals Involving Exponential and Logarithmic Functions

Recall some differentiation formulas

$$\frac{d}{dx}\left(e^x\right) = e^x$$

$$\frac{d}{dx}\left(a^x\right) = a^x \cdot \ln a$$

$$\left(\text{E.g. } \frac{d}{dx}\left(2^x\right) = 2^x \cdot \ln 2 \right)$$

$$\int e^x\,dx = e^x + C$$

$$\boxed{\int a^x\,dx = \frac{a^x}{\ln a} + C}$$

True b/c $\left(\frac{a^x}{\ln a}\right)' = \frac{1}{\cancel{\ln a}} \cdot a^x \cdot \cancel{\ln a} = a^x$

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$$\boxed{\begin{array}{l} ① \int e^x\,dx = e^x + C \\ ② \int a^x\,dx = \dfrac{a^x}{\ln a} + C \end{array}}$$

E.g. ① $\int e^{-x}\,dx$

u - substitution. $u = -x$. $du = -dx$

$$\int e^{-x}dx = -\int e^{\boxed{-x}}\underbrace{\boxed{(-dx)}}_{du} = -\int e^u\,du = -e^u + C = \boxed{-e^{-x} + C.}$$

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② $\int \boxed{e^x}\sqrt{\boxed{1+e^x}}\,\boxed{dx}$

Let $u = 1 + e^x$. Then $du = e^x\,dx$

$$\int \sqrt{\underbrace{\boxed{1+e^x}}_{u}} \cdot \underbrace{\boxed{e^x dx}}_{du} = \int \sqrt{u}\,du = \int u^{\frac{1}{2}}du = \frac{2u^{\frac{3}{2}}}{3} + C = \boxed{\frac{2}{3}\cdot\left(1+e^x\right)^{\frac{3}{2}} + C.}$$

③ $\int \boxed{x^3}\, e^{\boxed{x^4}}\,\boxed{dx}$

Let $u = x^4$. Then $du = 4x^3dx$

$$\frac{1}{4}\int e^{\boxed{x^4}\to u} \cdot \underbrace{\boxed{4x^3dx}}_{du} = \frac{1}{4}\cdot\int e^u\,du = \frac{1}{4}e^u + C = \boxed{\frac{1}{4}\cdot e^{x^4} + C}$$

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④ $\int_1^2 e^{\boxed{1-x}}\,dx$

Let $u = 1 - x$. Then $du = -dx$

$$-\int e^{\boxed{1-x}}\underbrace{\boxed{(-dx)}}_{du} = -\int e^u\,du = -e^u + C$$

$$= -e^{1-x}\Big|_1^2 = -e^{1-2} - \left(-e^{1-1}\right)$$

$$= -e^{-1} + e^0 = \boxed{-\frac{1}{e} + 1}$$

⑤ $\int_1^2 \frac{e^{\boxed{\frac{1}{x}}}}{x^2}\,dx$. Let $u = \frac{1}{x}$; $du = -\frac{1}{x^2}dx$

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$$-\int e^{\boxed{\frac{1}{x}}}\underbrace{\boxed{\left(-\frac{dx}{x^2}\right)}}_{du} = -\int e^u\,du = -e^u + C$$

$$= -e^{\frac{1}{x}}\Big|_1^2 = -e^{\frac{1}{2}} - \left(-e^{\frac{1}{1}}\right)$$

$$= -e^{\frac{1}{2}} + e. = e - \sqrt{e}.$$

⑥ $\int 2^x\,dx = \frac{2^x}{\ln 2} + C$

⑦ $\int 2^{\boxed{-x}}\,dx = -\int 2^{\boxed{-x}}\underbrace{\boxed{(-dx)}}_{du} = -\int 2^u\,du = -\frac{2^u}{\ln 2} + C = \boxed{\frac{-2^{-x}}{\ln 2} + C}$

Let $u = -x$; $du = -dx$

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Integrals that involve Log functions

Recall: $\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x}$.

$$\frac{d}{dx}\left(\log_a|x|\right) = \frac{1}{x\ln a}$$

$$\boxed{① \int \frac{dx}{x} = \ln|x| + C}$$

$$\boxed{② \int \ln x\,dx = x\ln x - x + C}$$

Proof: $(x\ln x - x)' = (x\ln x)' - 1$
$= 1\cdot\ln x + x\cdot\frac{1}{x} - 1 = \ln x + 1 - 1 = \ln x$

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