5.5. Integration by Substitution (u-substitution)

This is a method to find integrals of the form

$$\int f(\underbrace{g(x)}_{})\,\underbrace{g'(x)}_{}\,dx$$

both $g(x)$ and $g'(x)$ appear in the integrand.

E.g. $\int \underbrace{\sin}_{f}(\underbrace{x^3}_{g(x)})\cdot \underbrace{3x^2}_{g'(x)}\,dx$

u-substitution: $\int f(\underbrace{g(x)}_{u})\,\underbrace{g'(x)\,dx}_{du} \longrightarrow \boxed{\int f(u)\,du}$

let $u = g(x)$. $\frac{du}{dx} = g'(x)$; $du = g'(x)\,dx$

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$$\int \sin(\underbrace{x^3}_{u})\cdot \underbrace{3x^2\,dx}_{du}$$

let $u = x^3$. Then $du = 3x^2\,dx$

original integral in terms of x becomes

$$\int \sin(u)\,du = -\cos(u) + C.$$

$$= -\cos(x^3) + C.$$

$$\boxed{\int \sin(x^3)\cdot 3x^2\,dx = -\cos(x^3) + C}$$

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$$\int \sin(\underbrace{x^3}_{u})\cdot 9x^2\,dx$$

let $u = x^3$. $du = 3x^2\,dx$

$$\int \sin(x^3)\cdot 3\cdot 3x^2\,dx = 3\int \sin(\overset{u}{x^3})\cdot \overset{du}{3x^2\,dx}$$

$$= 3\cdot\int \sin(u)\,du = -3\cos(u) + C.$$

$$\frac{1}{3}\int \sin(\overset{u}{x^3})\,\overset{du}{3x^2\,dx} = \frac{1}{3}\int \sin(u)\,du = -\frac{1}{3}\cos(u) + c.$$

let $u = x^3$. $du = 3x^2\,dx$

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More examples of Integration by Substitution

E.g. 1. $\frac{1}{3}\int \frac{\overbrace{3\cdot x^2\,dx}^{du}}{\underbrace{x^3+5}_{u}}$

let $u = x^3+5$. $du = 3x^2\,dx$

$$\frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u| + C.$$

$$= \boxed{\frac{1}{3}\ln|x^3+5| + C}.$$

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$$-\int \underbrace{\cos^2 x}_{u^2}\,(\underbrace{-\sin x\,dx}_{du})$$

let $u = \cos x$; $du = -\sin x\,dx$

$$-\int u^2\,du = -\frac{u^3}{3} + C = \boxed{-\frac{\cos^3 x}{3} + C}$$

E.g. $\int z\sqrt{z^2-5}\,dz = \frac{1}{2}\int\sqrt{\underbrace{z^2-5}_{u}}\cdot \overbrace{2z\,dz}^{du}$

let $u = z^2-5$. $du = 2z\,dz$

$$= \frac{1}{2}\cdot\int\sqrt{u}\,du = \frac{1}{2}\int u^{\frac{1}{2}}\,du = \frac{1}{2}\cdot\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$= \frac{1}{3}u^{\frac{3}{2}} + C = \boxed{\frac{1}{3}(z^2-5)^{\frac{3}{2}} + C}$$

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E.g. $-\int \frac{\overbrace{-\sin t\,dt}^{du}}{\underbrace{\cos^3 t}_{u^3}}$

let $u = \cos t$. $du = -\sin t\,dt$

$$-\int \frac{du}{u^3} = -\int u^{-3}\,du = -\frac{u^{-2}}{-2} + C = \frac{1}{2}\cdot\frac{1}{u^2} + C$$

$$= \frac{1}{2\cos^2 t} + C = \boxed{\frac{1}{2}\sec^2 t + C}.$$

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