4.8. L'Hopital Rule

L'Hopital Rule for limits of the form $\frac{0}{0}$

E.g. $\lim_{x\to \boxed{3}} \frac{x^2-2x-3}{x^2-9}$ $\quad \frac{(3)^2-2\cdot 3-3}{(3)^2-9} = \boxed{\frac{0}{0}}$

$= \lim_{x\to \boxed{3}} \frac{2x-2}{2x} = \frac{2\cdot 3-2}{2\cdot 3} = \frac{4}{6} = \boxed{\frac{2}{3}}$

Factor: $\lim_{x\to 3} \frac{\cancel{(x-3)}(x+1)}{\cancel{(x-3)}(x+3)} = \lim_{x\to 3} \frac{x+1}{x+3} = \frac{4}{6} = \boxed{\frac{2}{3}}$.

Nov 7-4:18 PM

L'Hopital Rule for $\frac{0}{0}$ limits

We want to find $\lim_{x\to a} \frac{f(x)}{g(x)}$ where $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ ($\frac{0}{0}$ limit)

L'Hopital rule says that

$$\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}}$$

E.g. $\lim_{x\to \boxed{0}} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} \stackrel{\frac{0}{0}}{=} \lim_{x\to 0} \frac{\frac{1}{2\sqrt{1+x}} - \frac{1}{2\sqrt{1-x}}\cdot(-1)}{1}$

$= \lim_{x\to 0} \left(\frac{1}{2\sqrt{1+x}} + \frac{1}{2\sqrt{1-x}}\right) = \frac{1}{2} + \frac{1}{2} = \boxed{1}$.

Nov 7-4:35 PM

E.g. $\lim_{x\to 0} \frac{e^x - x - 1}{x^2} \stackrel{\frac{0}{0}}{=} \lim_{x\to 0} \frac{e^x - 1}{2x} \stackrel{\frac{0}{0}}{=}$

$\lim_{x\to 0} \frac{e^x}{2} = \boxed{\frac{1}{2}}$.

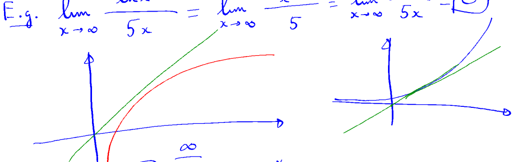
E.g. $\lim_{x\to 0} \frac{x}{\tan x} \stackrel{\frac{0}{0}}{=} \lim_{x\to 0} \frac{1}{\sec^2 x} = \boxed{1}$.

Nov 7-4:39 PM

L'Hopital Rule for limits of the form $\frac{\infty}{\infty}$

E.g. $\lim_{x\to\infty} \frac{2x+5}{7x+17} \stackrel{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{2}{7} = \boxed{\frac{2}{7}}$

E.g. $\lim_{x\to\infty} \frac{\ln x}{5x} \stackrel{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{5} = \lim_{x\to\infty} \frac{1}{5x} = \boxed{0}$



E.g. $\lim_{x\to\infty} \frac{\boxed{e^x}}{\boxed{x}} \stackrel{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{e^x}{1} = \lim_{x\to\infty} e^x = \boxed{\infty}$

Nov 7-4:43 PM

Limits of the form $\boxed{\frac{0}{0}\ ;\ \frac{\infty}{\infty}} \longrightarrow$ indeterminate forms

Other Indeterminate forms

$0\cdot\infty$ $\quad\left(\text{E.g. } \lim_{x\to 0^+} (\underbrace{x}_{0}\cdot\underbrace{\ln x}_{\infty})\right)$

$\infty-\infty$ $\quad\left(\text{E.g. } \lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right)\right)$

∞^0 $\quad\left(\text{E.g. } \lim_{x\to\infty} \underbrace{\boxed{x}}_{\to\infty}^{\boxed{\frac{1}{x}}\to 0}\right)$

1^∞ $\quad\left(\text{E.g. } \lim_{x\to\infty} \underbrace{\left(\boxed{1+\frac{1}{x}}\right)}_{\to 1}^{x\to\infty}\right)$

0^0 $\quad\left(\text{E.g. } \lim_{x\to 0} \underbrace{\boxed{x}}_{0}^{\boxed{\sin x}\to 0}\right)$

Idea: transform these forms to either $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form and apply L'Hopital Rule.

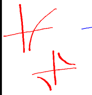
Nov 7-4:50 PM

Indeterminate form $\boxed{0}\cdot\infty$

E.g. $\lim_{x\to 0^+} (\boxed{x}\cdot\boxed{\ln x})$ $\quad 0\cdot(-\infty)$

Note: L'Hopital rule does not apply directly to this form)

$= \lim_{x\to 0^+} \frac{\boxed{\ln x}\to -\infty}{\frac{1}{x}\to\infty} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{1}{x}\cdot(-x^2)$



$= \lim_{x\to 0^+} (-x) = \boxed{0}$.

Nov 7-4:58 PM