3.7 - Rational Functions

$f(x) = \frac{x-7}{3x+5}$; $f(x) = \frac{x^2+10x-5}{3x^3+6x^2-x+11}$

A rational function is a function of the form

$f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials

Goals:
1. Find the domain, Vertical asymptotes, horizontal asymptotes, slant asymptotes, x-intercept, y-intercept of a rational function.
2. Find an equation for a rational function given its graph
3. Identify Removable Discontinuities of a rational function.
4. Solve application problems.

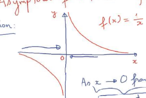
Domain of a Rational function.

$f(x) = \frac{4x}{5(x-1)(x-5)}$

Domain? $(-\infty,1) \cup (1,5) \cup (5,\infty)$

Vertical Asymptotes of a Rational function.

Arrow notation:



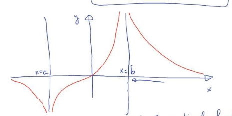
$f(x) = \frac{1}{x}$

As $\underbrace{x \to 0 \text{ from the right}}$, $f(x) \to \infty$

$\boxed{x \to 0^+,\ f(x) \to \infty}$

As $\underbrace{x \to 0 \text{ from the left}}$, $f(x) \to -\infty$

$\boxed{x \to 0^-,\ f(x) \to -\infty}$



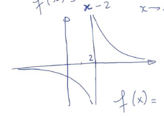
Definition of a vertical asymptote of a rational function.

We say that the vertical line $x = a$ is a vertical asymptote of the function $f(x)$ if $f(x) \to \infty$ or $-\infty$ as $x \to a^+$ or $x \to a^-$

Examples of finding V.A. of Rational functions

$f(x) = \frac{1}{x-2}$ $\boxed{x=2 \text{ is a V.A.}}$

$x \to 2^+$, $f(x) \to \infty$



$f(x) = \frac{x+4}{x-6}$

$x=6$ is a V.A.

$f(x) = \frac{2x+1}{(x-1)(x-2)(x-3)(x+4)}$

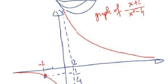
V.A. $x=1$; $x=2$; $x=3$; $x=-4$

$f(x) = \frac{x+2}{x^2-4} = \frac{\cancel{x+2}}{\boxed{x-2}(\cancel{x+2})} = \begin{cases} \boxed{\frac{1}{x-2}} & \text{if } x \neq -2 \\ \text{undefined} & \text{if } \boxed{x=-2} \end{cases}$

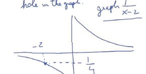
$x = 2$ is a V.A.

hole in the graph.

graph of $\frac{x+2}{x^2-4}$



graph $\frac{1}{x-2}$



Steps for finding the V.A.(s) of a rational function $f(x) = \frac{p(x)}{q(x)}$,

1. Factor the numerator $p(x)$ and the denominator $q(x)$ completely.
2. Cancel the common factor(s) of $p(x)$ and $q(x)$.
3. The values of x which make the denominator of the simplified function 0 will correspond to the vertical asymptotes.

E.g. $f(x) = \boxed{\frac{x^2+6x+5}{x^2-25}} \leftarrow \frac{0}{0}$

Find the V.A. of f

$f(x) = \frac{\cancel{(x+5)}(x+1)}{\cancel{(x+5)}(x-5)}$ (Step 1)

$\boxed{\frac{x+1}{x-5}}$ (Step 2)

$x = 5$ is a V.A. (Step 3)

$x=-5$ corresponds to a hole in the graph.

y-coordinate of the hole is $\frac{-5+1}{-5-5} = \frac{-4}{-10} = \frac{2}{5}$

$\left(-5, \frac{2}{5}\right)$ hole in the graph.

$f(x) = \frac{3x-12}{x^3-16x} = \frac{3(x-4)}{x(x^2-16)} = \frac{3\cancel{(x-4)}}{x\cdot\cancel{(x-4)}(x+4)}$

Find V.A.(s) and hole(s) of this function

$x=0$; $x=-4$ are the V.A.

$x=4$; $\frac{3}{x(x+4)} = \frac{3}{4\cdot 8} = \frac{3}{32}$; $\left(4, \frac{3}{32}\right)$ hole in graph

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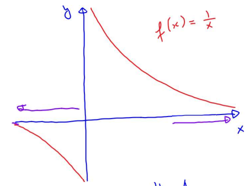
Horizontal Asymptotes



H.A. $y = 0$

In arrow notation:

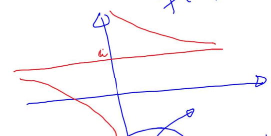
As $x \to \infty$, $f(x) \to 0$

As $x \to -\infty$, $f(x) \to 0$

Definition of a H.A.

The line $y = a$ is a H.A. of the function $f(x)$ if

$f(x) \longrightarrow a$ whenever $x \to \infty$ or $x \to -\infty$.



$f(x) = \frac{1x^2}{3x^2 + 7}$ Find the H.A.

$\boxed{\approx} \frac{\cancel{x^2}}{3\cancel{x^2}} = \frac{1}{3}$ $\boxed{y = \frac{1}{3}}$ is the H.A.

As $x \to \infty$; $f(x) \longrightarrow \frac{1}{3}$.

$f(x) = \frac{2x^2}{x^4 + 6}$ Find the H.A.

$\approx \frac{2x^2}{x^4} \approx \frac{2}{x^2} \approx 0$. As $x \to \infty$; $f(x) \to 0$

$\boxed{y = 0}$ is the H.A.

$f(x) = \boxed{\frac{x^2}{3x + 5}} \approx \frac{x^2}{3x} = \frac{x}{3} \to \infty$. There's no H.A.

In general, $f(x) = \frac{p(x)}{q(x)}$

① If degree $p(x)$ = degree $q(x)$; then $y = \frac{a}{b}$ is the H.A.

where a is the leading coefficient of $p(x)$ and b is the leading coefficient of $q(x)$.

② If degree $p(x) <$ degree $q(x)$, then $y = 0$ is the H.A.

③ If degree $p(x) >$ degree $q(x)$, then there is no H.A.

Slant Asymptotes

Note: The only situation for which $f(x) = \frac{p(x)}{q(x)}$ has a slant asymptote is when degree $p(x) = 1 +$ degree $q(x)$

E.g. $f(x) = \frac{2x^2 + x - 1}{x - 4}$

How to find a S.A.? Do a Long Division.

$$
\begin{array}{r l}
 & \boxed{2x + 9} \longrightarrow \text{quotient} \\
\boxed{x} - 4 \,\big)\! & \boxed{2x^2} + x - 1 \\
 & -(2x^2 - 8x) \\
 & \boxed{\boxed{9x} - 1} \\
 & -(9x - 36) \\
 & \boxed{35} \longrightarrow \text{Remainder.}
\end{array}
$$

$\frac{2x^2}{x} = 2x$

$\frac{9x}{x} = 9$

The equation $y = 2x + 9$ is the equation of the S.A.