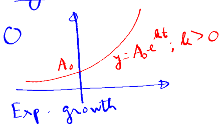
4.7 Exponential and Logarithmic Models

Goal: Solve application problems that involve exponential and logarithmic functions.

* Model Exponential growth and decay.

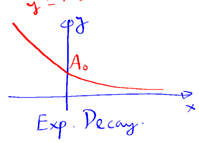
$$y = A_0 e^{kt}$$

$k > 0$



Exp. growth

$y = A_0 e^{kt}$; $k < 0$.



Exp. Decay.

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E.g. 1:

A population of bacteria doubles every hour.

If the culture started with 10 bacteria, find the function which gives the # of bacteria at time t.

$$\boxed{y = A_0 e^{kt}}$$

$y = 10 e^{kt}$.

$\cancel{k = 2}$. $\cancel{y = 10e^{2t}}$

$\cancel{t = 1;\ y = 10e^2}$

$t = 1$; $y = 20$

$20 = 10 \cdot e^k$; $e^k = 2$

$$\boxed{k = \ln(2)}$$

$$\boxed{y = 10 e^{\ln(2) t}}$$

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Half-Life Problems

E.g. The half-life of Carbon-14 is 5730 years. Express the amount of carbon-14 as a function of time t.

$$\boxed{y = A_0 e^{\frac{\ln(0.5)}{5730} t}}$$

$y = \boxed{A_0} e^{kt}$;

Find k. At $t = 5730$; $y = \frac{A_0}{2}$

$\frac{A_0}{2} = A_0 e^{k(5730)}$

$\frac{1}{2} = e^{k(5730)}$; $\ln \frac{1}{2} = \ln e^{5730k}$

$\ln \frac{1}{2} = 5730k$; $\boxed{k = \frac{\ln(\frac{1}{2})}{5730}}$

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E.g. A bone fragment that contains 20% of its original Carbon-14. How old is the bone?

$$\boxed{y = \boxed{A_0} e^{\frac{\ln(0.5)}{5730} t}}$$

$\frac{\cancel{A_0}}{5} = \cancel{A_0} e^{\frac{\ln(0.5)}{5730} t}$

$\frac{1}{5} = e^{\frac{\ln(0.5)}{5730} t}$; $\ln \frac{1}{5} = \cancel{\ln e}$

$\ln \frac{1}{5} = \boxed{\frac{\ln(0.5)}{5730}} t$

$\frac{\ln(0.5)}{5730} t$

$t = \dfrac{\ln(\frac{1}{5})}{\frac{\ln(0.5)}{5730}}$

≈ 13304 years old.

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E.g. Cesium-137 has a half-life of about 30 years. If we begin with 200 mg of cesium-137, will it take more or less than 230 years until only 1 mg remains?

$$y = A_0 e^{kt}$$

Half life $= 30$, we have

$\frac{\cancel{A_0}}{2} = \cancel{A_0} \cdot e^{30k}$

$\frac{1}{2} = e^{30k}$; $\frac{\ln(\frac{1}{2})}{30} = k$.

$\frac{1}{2} = e^{\frac{\ln(0.5)}{30} t}$

$y = A_0 \cdot e^{\frac{\ln(0.5)}{30} t}$

$1 = 200 \cdot e^{\frac{\ln(0.5)}{30} t}$; $\frac{1}{200} = e^{\frac{\ln(0.5)}{30} t}$

$\frac{1}{200} = \frac{\ln(0.5)}{30} t$

$t = \dfrac{\ln(\frac{1}{200})}{\frac{\ln(0.5)}{30}}$

$t \approx 229$ years

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Newton's Law of Cooling

$$\boxed{T(t) = Ae^{kt} + T_s}$$

$T(t)$: temperature of the object at time t.

A: difference between the initial temperature of the object and the surroundings

k: constant, the continuous rate of cooling of the object.

T_s: temperature of surrounding air.

E.g. Use Newton's Law of Cooling.

A cheesecake is taken out of the oven with a temperature 165°F. It is placed into a 35°F refrigerator.

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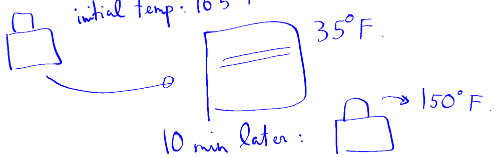
After 10 minutes the cheesecake has cooled to 150°F. If we must wait until the cheesecake has cooled to 70°F before we eat it, how long will we have to wait?

initial temp: 165°F



10 min later:

$A = 165 - 35 = 130$

$T(t) = 130e^{kt} + 35$

When $t = 10$; $T(10) = 150$ Therefore,

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$150 = 130e^{10k} + 35$

$115 = 130e^{10k}$; $k = \dfrac{\ln\left(\frac{115}{130}\right)}{10}$.

$T(t) = 130e^{\frac{\ln\left(\frac{115}{130}\right)}{10}t} + 35$

Want: find t such that $T(t) = 70$.

$70 = 130e^{\frac{\ln\left(\frac{115}{130}\right)}{10}t} + 35$

$35 = 130e^{\frac{\ln\left(\frac{115}{130}\right)t}{10}}$

$\dfrac{35}{130} = e^{\boxed{\frac{\ln\left(\frac{115}{130}\right)t}{10}}}$; $t = \dfrac{\ln\left(\frac{35}{130}\right)}{\frac{\ln\left(\frac{115}{130}\right)}{10}} \approx 107$ min

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Ex: A pitcher of water at 40°F is placed to a 70°F room. One hour later, the temperature has risen into 45°F.

How long will it take for the temperature to rise to 60°F?

$T(t) = -30e^{kt} + 70$

$45 = -30e^{k} + 70$

$-25 = -30e^{k}$; $\dfrac{25}{30} = e^{k}$; $k = \ln\left(\frac{25}{30}\right)$

$T(t) = -30e^{\ln\left(\frac{25}{30}\right)t} + 70$; $60 = -30e^{\ln\left(\frac{25}{30}\right)t} + 70$

$-10 = -30e^{\ln\left(\frac{25}{30}\right)t}$

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$\dfrac{1}{3} = e^{\ln\left(\frac{25}{30}\right)t}$

$t = \dfrac{\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{25}{30}\right)} \approx 6$ (hours).

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