Problems from practice Exam

21 a) $\lim_{x \to -1} \frac{2x^2+3x+1}{x^2-2x-3} \cdot \left(\frac{0}{0}\right)$

$= \lim_{x\to -1} \frac{(2x+1)\cancel{(x+1)}}{(x-3)\cancel{(x+1)}} = \lim_{x \to -1} \frac{2x+1}{x-3}$

$= \frac{2(-1)+1}{-1-3} = \frac{-2+1}{-4} = \frac{-1}{-4} = \boxed{\frac{1}{4}}.$

Sep 21-4:10 PM

$\lim_{t \to 0} \frac{\boxed{\sqrt{1+t} - \sqrt{1-t}}}{t} \quad \left(\frac{0}{0}\right) \quad \boxed{(A-B)(A+B) = A^2 - B^2.}$

$= \lim_{t\to 0} \frac{(\sqrt{1+t}) - (\sqrt{1-t})}{t} \cdot \frac{(\sqrt{1+t} + \sqrt{1-t})}{\sqrt{1+t}+\sqrt{1-t}}.$

$= \lim_{t \to 0} \frac{1+t-(1-t)}{t\cdot(\sqrt{1+t}+\sqrt{1-t})}$

$= \lim_{t\to 0} \frac{\cancel{1}+t \cancel{-1}+t}{t\cdot(\sqrt{1+t}+\sqrt{1-t})}$

Sep 21-4:17 PM

$= \lim_{t\to 0} \frac{2\cancel{t}}{\cancel{t}(\sqrt{1+t}+\sqrt{1-t})}$

$= \lim_{t \to 0} \frac{2}{\sqrt{1+t}+\sqrt{1-t}}$

$= \frac{2}{\sqrt{1}+\sqrt{1}} = \frac{2}{2} = \boxed{1}.$

Sep 21-4:21 PM

22 (a) $f(x) = \begin{cases} \boxed{\cos(x)} & \text{if } x<0 \\ \boxed{0 \text{ if } x=0} & \\ \boxed{1-x^2} & \text{if } x>0. \end{cases}$

Explain why f is discontinuous at $a=0$.

(1) $\boxed{\lim_{x\to 0} f(x) = 1}$ ✓

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0} (\cos x) = \cos(0) = \boxed{1}.$

$\lim_{x\to 0^+} f(x) = \lim_{x \to 0} (1-x^2) = \boxed{1}$

Sep 21-4:23 PM

(2) $f(0)$ must exist

$f(0)=0$ ✓

(3) $\lim_{x\to 0} f(x) \neq f(0)$

$\underbrace{\lim_{x\to 0} f(x)}_{1} \neq \underbrace{f(0)}_{0}$

So, f is discontinuous at 0. This is because $\lim_{x\to 0} f(x) \neq f(0)$.

Sep 21-4:27 PM

22 (b) $f(x) = \begin{cases} \boxed{x^4 \sin\left(\frac{1}{x}\right)} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

Explain why f is continuous on $(-\infty, \infty)$

If $x \neq 0$; $f(x) = x^4 \sin\left(\frac{1}{x}\right)$

Any # that is nonzero belongs to the domain of f.

f is continuous at any point $x \neq 0$.

Show that f is continuous at $x=0$.

Sep 21-4:31 PM

Need: $\lim_{x\to 0} f(x) = \boxed{f(0)}$

$f(0) = 0.$

What about $\lim_{x\to 0} f(x)$

$\lim_{x\to 0} f(x) = \boxed{\lim_{x\to 0} x^4 \boxed{\sin\left(\frac{1}{x}\right)}}.$

$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$ for all $x \neq 0.$

$\boxed{-x^4} \leq \boxed{x^4 \sin\left(\frac{1}{x}\right)} \leq \boxed{x^4}.$

0

Sep 21-4:34 PM

$\lim_{x\to 0}(-x^4) = 0$; $\lim_{x\to 0}(x^4) = 0$

By the Squeeze Theorem

$$\lim_{x\to 0} x^4 \sin\left(\frac{1}{x}\right) = 0$$

Sep 21-4:37 PM

23 (a) $f(x) = \boxed{4x - 3x^2}$

Find $f'(2)$ using the definition of the derivative.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$\boxed{f'(2)} = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h\to 0} \frac{\left[4(2+h) - 3(2+h)^2\right] - (-4)}{h}$$

Sep 21-4:38 PM

$$= \lim_{h\to 0} \frac{8 + 4h - 3(4 + 4h + h^2) + 4}{h}$$

$$= \lim_{h\to 0} \frac{8 + 4h \cancel{-12} - 12h - 3h^2 \cancel{+4}}{h}$$

$$= \lim_{h\to 0} \frac{-8h - 3h^2}{h} = \lim_{h\to 0} \frac{\cancel{h}(-8 - 3h)}{\cancel{h}}$$

$$= \lim_{h\to 0}(-8 - 3h) = \boxed{-8}.$$

$$\boxed{f'(2) = -8}.$$

Sep 21-4:42 PM

Find the equation of the tangent line to the graph of f at the point where $x = 2$.

Slope of tangent line to the graph of f at $x = 2$

$= f'(2) = \boxed{-8.}$

Point: $(2, -4)$

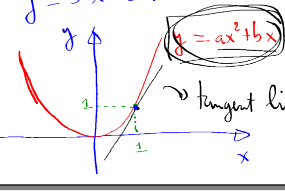
Point-Slope equation: $y - (-4) = -8(x - 2)$

$\boxed{y = -8x + 12}$

Sep 21-4:44 PM

24 (a)

Find a and b such that the equation to the tangent line at $(1, 1)$ to the parabola with equation $y = ax^2 + bx$ has equation $y = 3x - 2$.



tangent line has equation $\boxed{y = 3x - 2}$

Slope = 3

Sep 21-4:47 PM

$\boxed{y = ax^2 + bx}$

$y' = 2ax + b.$

At $x = 1$; $y'(1) = 2a + b$

$\boxed{2a + b = 3}$

Since $(1,1)$ belongs to the parabola,

$\boxed{1 = a + b}$; $b = 1 - a$

$2a + (1 - a) = 3$

$a + 1 = 3$; $\boxed{a = 2}$; $\boxed{b = -1}$.

Sep 21-4:51 PM

24 (b)

$$f(x) = \begin{cases} \boxed{x^2} & \text{if } x \le 2 \\ \boxed{mx + b} & \text{if } x > 2. \end{cases}$$

Find m, b such that f is differentiable everywhere.

If $x < 2$; f is differentiable.

If $x > 2$; f is differentiable

Find m, b such that f is differentiable at $x = 2$.

Sep 21-4:53 PM

$\boxed{f'(2)} = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$

Want: $f'(2)$ exist

* $\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$ exists.

$\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}}$

$= \lim_{x \to 2} (x + 2) = \boxed{4}$

$\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = \boxed{\lim_{x \to 2} \frac{(mx + b) - 4}{x - 2}} = \frac{2m + b - 4}{0}$

Sep 21-4:56 PM

If $2m + b - 4 \neq 0$, then the right is either ∞ or $-\infty$ and it would be different from the left limit.

Hence, $2m + b - 4 = 0$

$\boxed{2m + b = 4}$; $\boxed{b = 4 - 2m}$

$\lim_{x \to 2} \frac{mx + b - 4}{x - 2} = \lim_{x \to 2} \frac{mx + \cancel{4} - 2m - \cancel{4}}{x - 2}$

$= \lim_{x \to 2} \frac{m\cancel{(x - 2)}}{\cancel{x - 2}} = \lim_{x \to 2} (m) = \boxed{m}$

So, $\boxed{m = 4}$ $\boxed{b = -4}$

Sep 21-5:00 PM