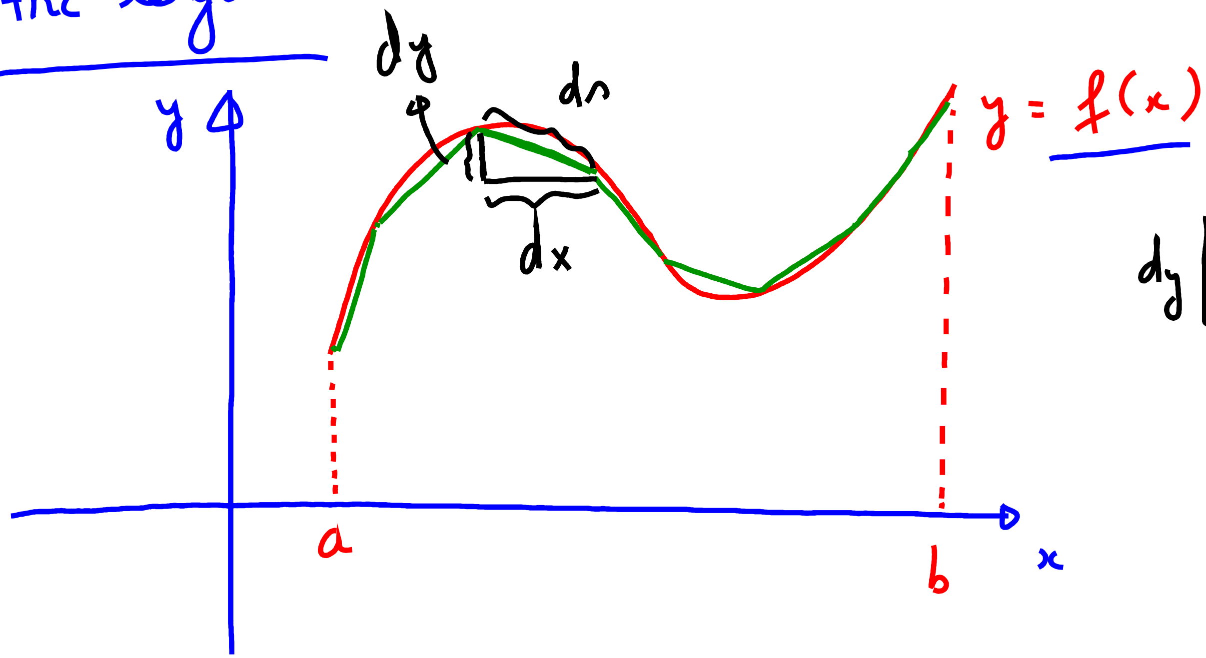
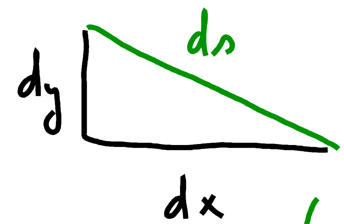
2.4. Arc length and Surface area.

Arc length





$$(ds)^2 = (dx)^2 + (dy)^2$$

Find the length of this curve from $x = a$ to $x = b$

Idea: divide the curve into small straight line segments, find the length of these line segments and add them up using integral.

$$(ds)^2 = (dx)^2 + (dy)^2$$

$$ds = \sqrt{\boxed{(dx)^2 + (dy)^2}}$$

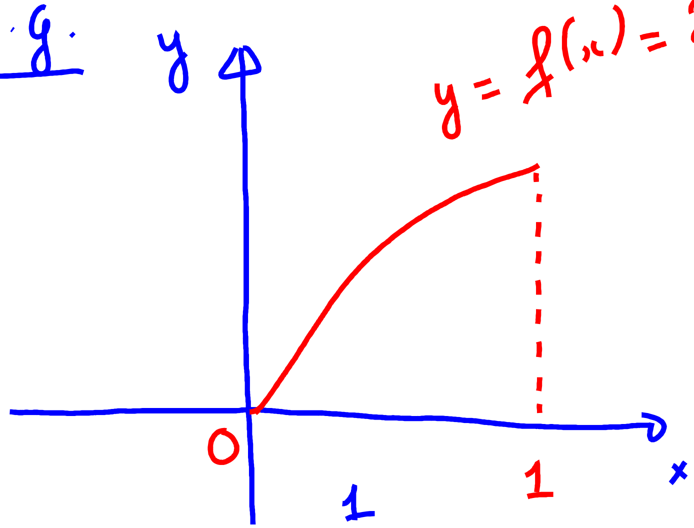
$$ds = \sqrt{(dx)^2\left[1 + \frac{(dy)^2}{(dx)^2}\right]} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$ds = \sqrt{1 + [f'(x)]^2}\cdot dx$$

$$\Longrightarrow \text{length of curve} = \int_a^b ds = \boxed{\int_a^b \sqrt{1 + [f'(x)]^2}\, dx}$$

E.g.

$y = f(x) = 2x^{3/2}$



Find the length of the curve $y = f(x) = 2x^{3/2}$; $0 \le x \le 1$

$$f'(x) = \cancel{2} \cdot \frac{3}{\cancel{2}} x^{\frac{1}{2}} = \boxed{3 \cdot x^{\frac{1}{2}}}$$

$$\text{length} = \int_0^1 \sqrt{1 + [f'(x)]^2}\, dx$$

$$= \int_0^1 \sqrt{1 + \left[3x^{\frac{1}{2}}\right]^2}\, dx = \frac{1}{9} \int_0^1 \sqrt{1 + 9x}\; \boxed{9\, dx}$$

du

u - sub $\boxed{u = 1 + 9x}$; $du = 9\,dx$

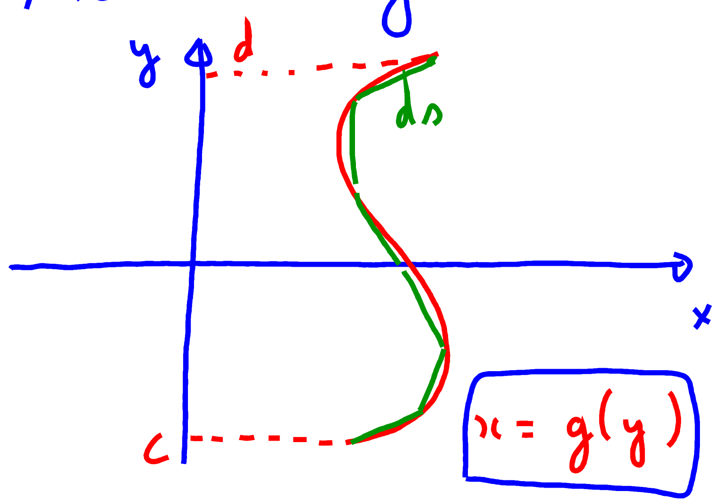
$$\boxed{\frac{1}{9} \int_1^{10} \sqrt{u}\, du}$$

When $x = 0$; $u = 1$; when $x = 1$; $u = 10$

$$\frac{1}{9}\int_1^{10}\sqrt{u}\,du = \frac{1}{9}\int_1^{10} u^{1/2}\,du = \frac{1}{9}\cdot\frac{2}{3}\cdot u^{3/2}\Big|_1^{10}$$

$$= \frac{2}{27}\cdot\left[(10)^{\frac{3}{2}} - 1\right] = \boxed{\frac{2}{27}\left[\sqrt{1000} - 1\right]}$$

More arc lengths.



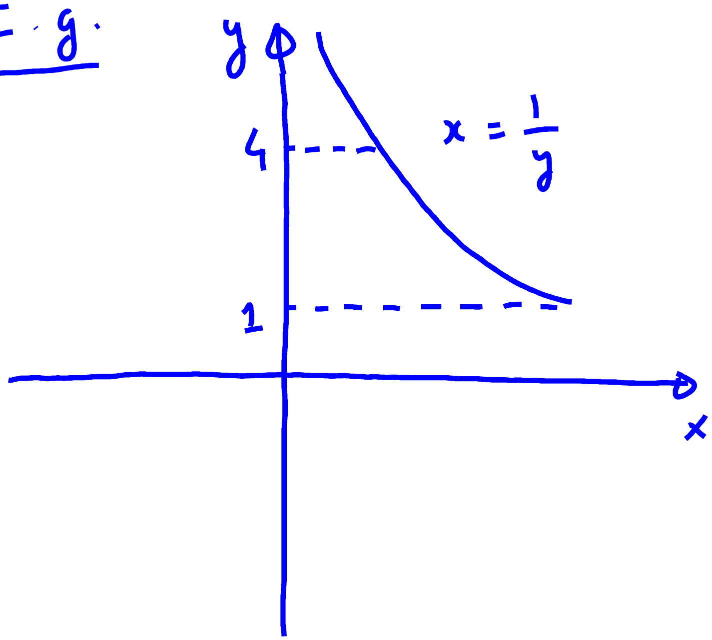
$$ds = \sqrt{(dx)^2 + (dy)^2}$$

$$= \sqrt{(dy)^2\left[\left(\frac{dx}{dy}\right)^2 + 1\right]} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$

$$ds = \sqrt{1 + [g'(y)]^2}\,dy$$

$$\text{length} = \int_c^d ds = \int_c^d \sqrt{1 + [g'(y)]^2}\,dy$$

E.g.



Find the length of the curve

$x = \frac{1}{y}$; $1 \le y \le 4$.

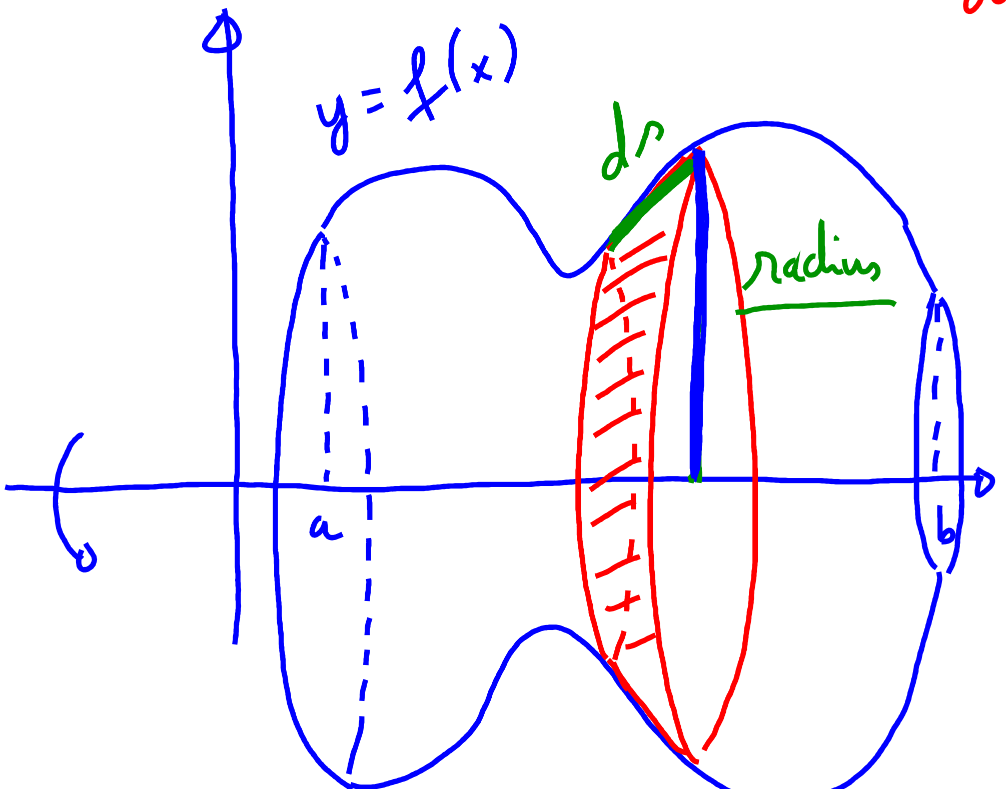
$$\text{Length} = \int_1^4 \sqrt{1 + \left(-\frac{1}{y^2}\right)^2}\, dy$$

$$= \int_1^4 \sqrt{1 + \frac{1}{y^4}}\, dy$$

$$= \int_1^4 \sqrt{\frac{y^4+1}{y^4}}\, dy \approx \boxed{21.0924}$$

Surface area of surface of revolution

divide surface into small patches



$$ds = \sqrt{1 + [f'(x)]^2}\, dx$$

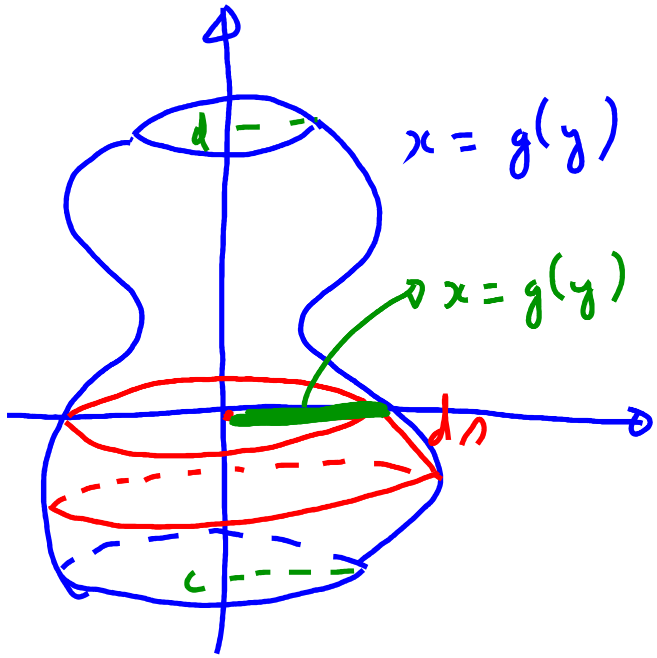
Area of a small patch

$$= 2\pi \cdot \underbrace{(\text{radius of rotation})} \cdot ds$$

$$= 2\pi \cdot f(x) \cdot \sqrt{1 + [f'(x)]^2}\, dx$$

Total surface area

$$= \boxed{2\pi \int_a^b f(x) \cdot \sqrt{1 + [f'(x)]^2}\, dx}$$



$$ds = \sqrt{1+[g'(y)]^2}\,dy$$

Area of a patch =

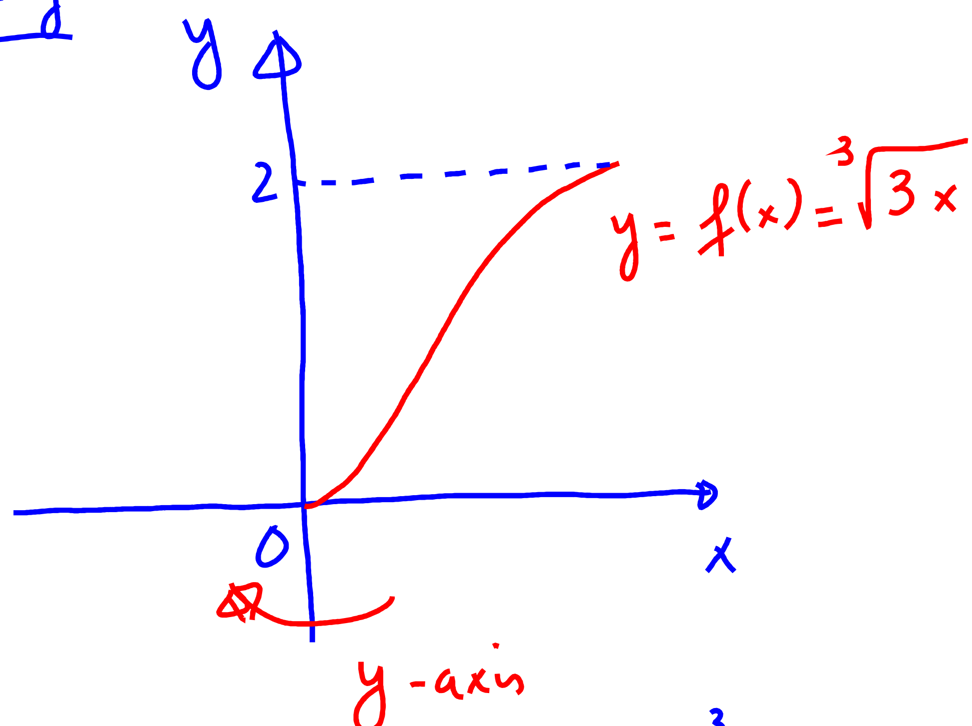
$2\pi \cdot$ radius $\cdot\ ds$

$$= 2\pi \cdot g(y) \cdot \sqrt{1+[g'(y)]^2}\,dy$$

Total surface area

$$= \int_c^d 2\pi\, g(y) \sqrt{1+[g'(y)]^2}\,dy.$$

E.g.



Find the surface area of the surface obtained by rotating the curve about y-axis.

$y = \sqrt[3]{3x}$; $\quad x = \dfrac{y^3}{3} = g(y)$

$$S = 2\pi \int_0^2 \frac{y^3}{3} \cdot \sqrt{1 + (y^2)^2}\, dy = 2\pi \cdot \frac{1}{4} \int_0^2 \frac{4y^3}{3} \cdot \sqrt{1 + y^4}\, dy$$

$u = 1 + y^4$. $\quad du = 4y^3\, dy$

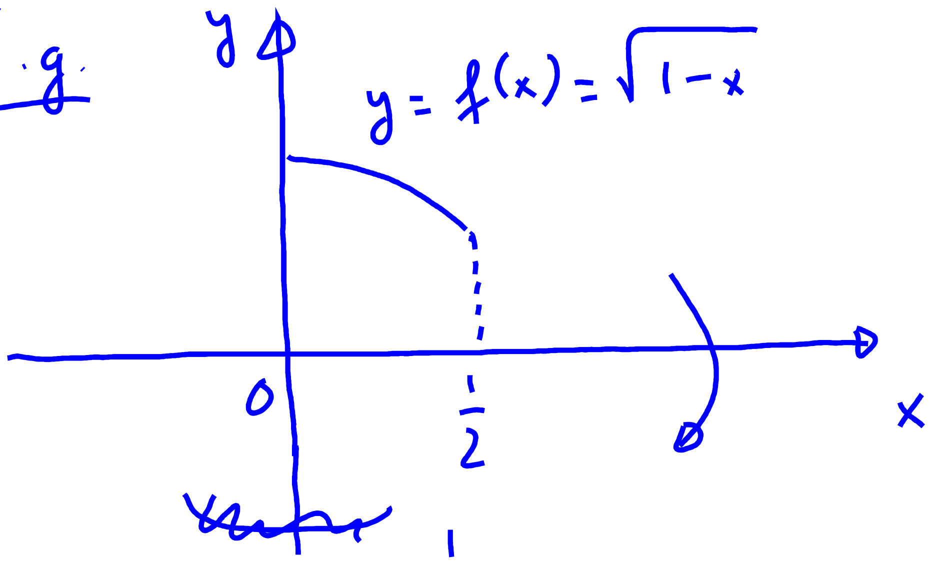
$y = 0$; $u = 1$; $y = 2$; $u = 17$

$$\frac{\pi}{6} \int_1^{17} \sqrt{u}\, du$$

$$\frac{\pi}{6}\int_{1}^{17} u^{\frac{1}{2}}\,du = \frac{\pi}{6}\cdot\frac{2}{3}\,u^{3/2}\Big|_{1}^{17}$$

$$= \frac{\pi}{9}\cdot\left((17)^{3/2} - 1\right)$$

E.g



Find the surface area of the surface obtained by rotating this curve $y = \sqrt{1-x}$; $0 \le x \le \frac{1}{2}$ about x-axis.

$$S = 2\pi \int_{0}^{\frac{1}{2}} f(x)\cdot\sqrt{1+[f'(x)]^2}\,dx \quad ;$$

$$f(x) = \sqrt{1-x}$$

$$f'(x) = \frac{-1}{2\sqrt{1-x}}.$$

$$S = 2\pi \int_0^{1/2} \sqrt{1-x} \cdot \sqrt{1 + \left(\frac{-1}{2\sqrt{1-x}}\right)^2}\, dx$$

$$= 2\pi \cdot \int_0^{1/2} \sqrt{1-x} \cdot \sqrt{1 + \frac{1}{4(1-x)}}\, dx$$

$$= 2\pi \int_0^{1/2} \sqrt{1-x} \cdot \sqrt{\frac{4-4x+1}{4(1-x)}}\, dx$$

$$= 2\pi \int_0^{1/2} \cancel{\sqrt{1-x}} \cdot \sqrt{\frac{5-4x}{4\cancel{(1-x)}}}\, dx = \cancel{2}\pi \int_0^{1/2} \frac{\sqrt{5-4x}}{\cancel{2}}\, dx$$

$$= \pi \left(-\frac{1}{4}\right) \int_0^{1/2} \sqrt{5-4x}\, \underbrace{-4\, dx}_{du}$$

$u = 5 - 4x$; $du = -4\, dx$

$x = 0$; $u = 5$; $x = \frac{1}{2}$; $u = 3$

$$-\frac{\pi}{4}\int_5^3 \sqrt{u}\,du = \frac{\pi}{4}\cdot\int_3^5 \sqrt{u}\,du$$

$$= \frac{\pi}{4}\cdot\frac{2}{3}\cdot u^{\frac{3}{2}}\Big|_3^5 = \frac{\pi}{6}\cdot\left((5)^{\frac{3}{2}} - (3)^{\frac{3}{2}}\right)$$

$$= \frac{\pi}{6}\cdot\left(\sqrt{125} - \sqrt{27}\right)$$

$$= \boxed{\frac{\pi}{6}\cdot\left(5\sqrt{5} - 3\sqrt{3}\right)}.$$