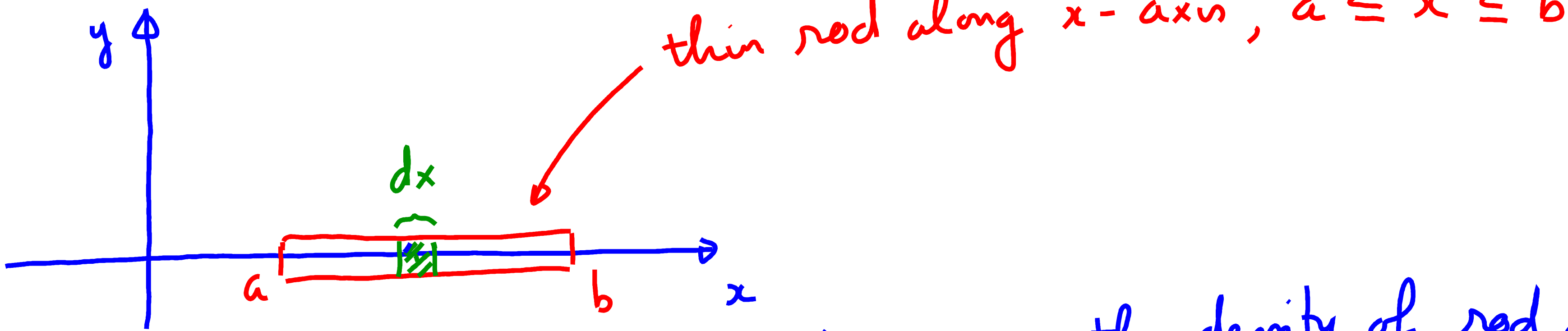
2.5. Physical Applications of Integrals

(1) Mass and density.



$\rho(x)$: density function. $\rho(x)$ gives us the density of rod at the point x. Question: $m_{rod} = ?$

density = mass per unit volume / area / length

dx : small segment of the rod. Density is almost constant throughout dx, given by $\rho(x)$

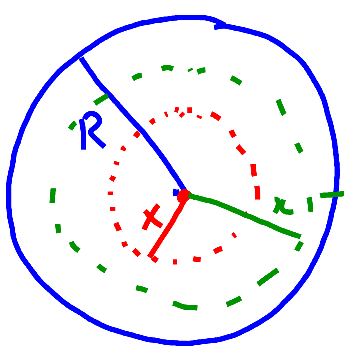
mass of the segment dx = $\rho(x)\,dx$

mass of the whole rod = $\sum$ mass of these segments

$$= \int_a^b \rho(x)\,dx$$

$$\boxed{m_{rod} = \int_a^b \rho(x)\,dx}$$

* Circular object

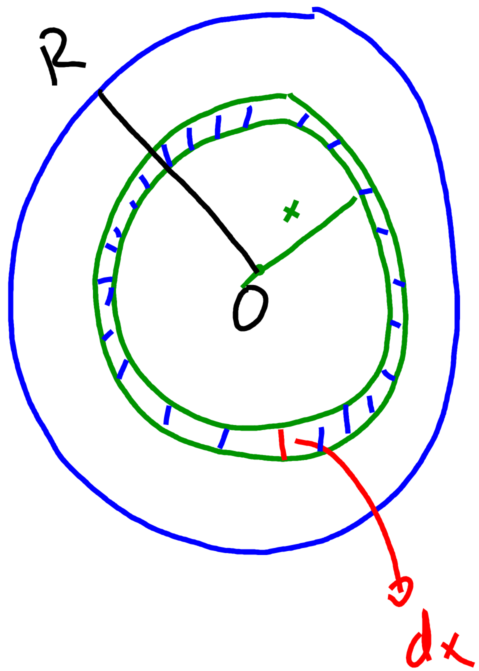


disk of radius R

$\rho(x)$: radical density.

every point on this circle of radius x has density $\rho(x)$

$$m_{disk} = \,?$$



Take a really "thin" washer with thickness dx

Area of washer : = (circumference) . (thickness)

$= 2\pi x \cdot dx$

mass of washer = (area) (density)

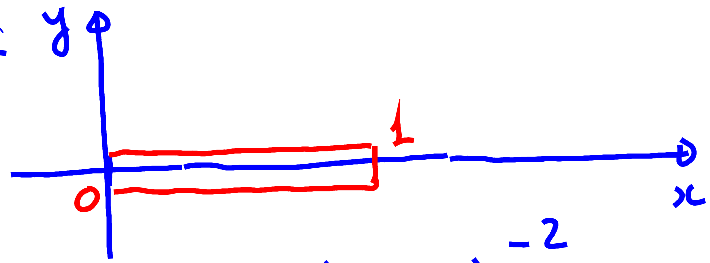
$= 2\pi x \rho(x) dx$

mass of the whole plate $= \int_0^R 2\pi x \rho(x) dx$

$$\boxed{m_{disk} = 2\pi \int_0^R x \rho(x) dx}$$

E.g.

(a)



$$\rho(x) = 5(x+2)^{-2}$$

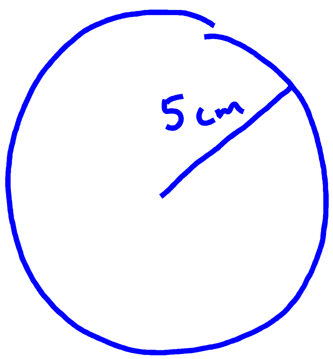
Q: m_{rod}.

$$\int_0^1 5(x+2)^{-2}\,dx \qquad u = x+2, \quad du = dx$$

$$5\int_2^3 u^{-2}\,du = 5\cdot\frac{u^{-1}}{-1}\Big|_2^3$$

$$-\frac{5}{u}\Big|_2^3 = -\frac{5}{3} + \frac{5}{2} = \boxed{\frac{5}{6}}$$

(b)



Radical density

$$\rho(x) = e^{-x^2} \ g/cm^2$$

Q: m_{disk}.

$$m_{disk} = \int_0^5 2\pi x\,\rho(x)\,dx$$

$$= 2\pi\left(-\tfrac{1}{2}\right)(-2)\int_0^5 \boxed{x}\, e^{-x^2}\,\boxed{dx}$$

$$u = -x^2\,; \quad du = -2x\,dx$$

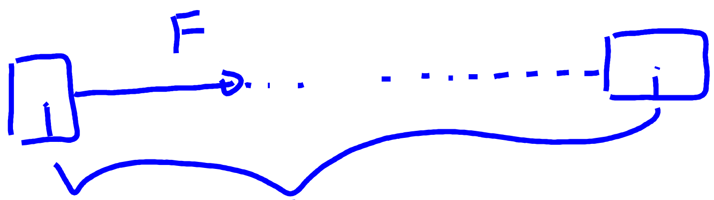
$$-\pi\int e^u\,du = -\pi e^u = -\pi e^{-x^2}\Big|_0^5$$

$$= -\pi\cdot e^{-25} + \underbrace{\pi e^0}_{\pi}$$

$$= \boxed{\pi\left(1 - e^{-25}\right)}$$

② Work done by a variable force.

F : constant force.



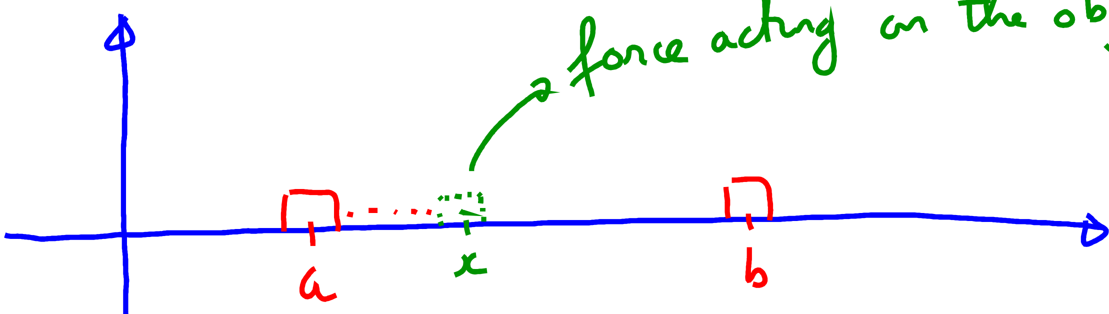
d = distance object traveled

$$W = (\text{Force}) \times (\text{Distance}) = F \cdot d.$$

(work)

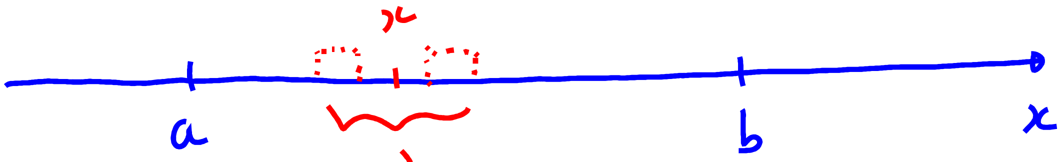
What if the force is variable ?

force acting on the object at the location x is given by $f(x)$



$\longrightarrow$ force is changing (given by $f(x)$)

Work done = ?



small distance = dx

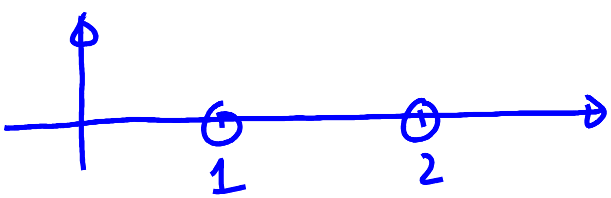
over this small distance $\longrightarrow$ force = constant = $f(x)$

work required to move the object this distance dx is : $w_{dx} = f(x)\,dx$.

total work required to move the object from a to b

$$\boxed{W = \int_a^b f(x)\,dx}$$

E.g.



variable force acting on the object is :

$$f(x) = x^2 + 2x$$

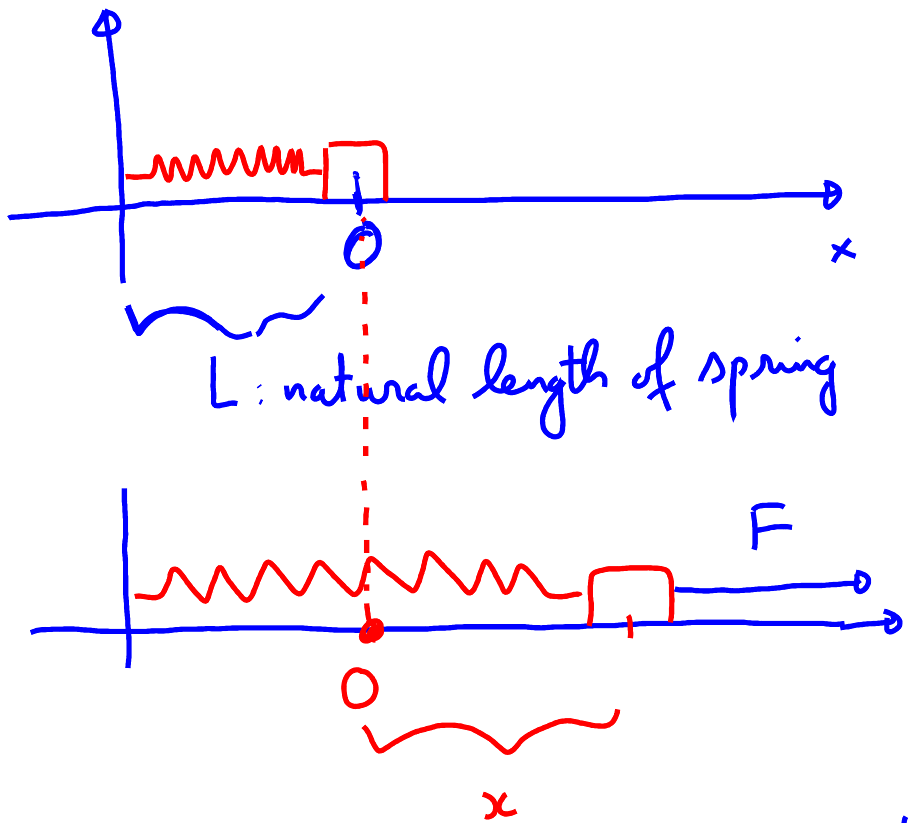
$W = ?$

$$W = \int_1^2 (x^2+2x)\,dx = \left(\frac{x^3}{3} + x^2\right)\Big|_1^2$$

$$= \dots$$

* Work done in stretching a spring.

Review of Hooke's Law in physics.



natural position of spring

Spring in stretched position

it is being stretched to a point x on x-axis

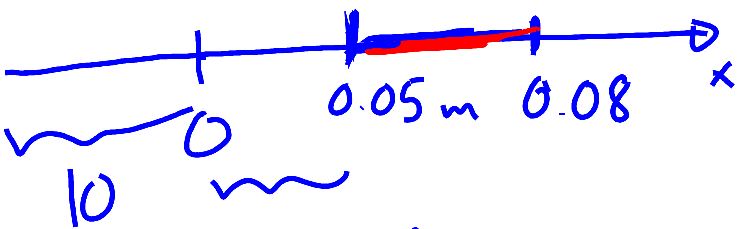
F : force required to maintain the spring in this stretched position

Hooke's law : $\boxed{F = kx}$ (k : spring constant, Hooke's constant)

E.g. Spring has natural length $L = 10$ cm.
A force of 40 N is required to hold the spring when it is stretched from its natural length to a length of 15 cm.

Q: Find the work done in stretching the spring from 15 cm to 18 cm?

$$\text{Force} = k \cdot x$$



$$40 = k \cdot (0.05)$$

$$\text{Hooke's constant} = k = \frac{40}{0.05} = 800$$

Force of 40 N

$$\boxed{f(x) = 800x}$$

$$W = \int_{0.05}^{0.08} f(x)\,dx = \int_{0.05}^{0.08} 800x\,dx = 800 \cdot \frac{x^2}{2} \Bigg|_{0.05}^{0.08} = \ldots$$