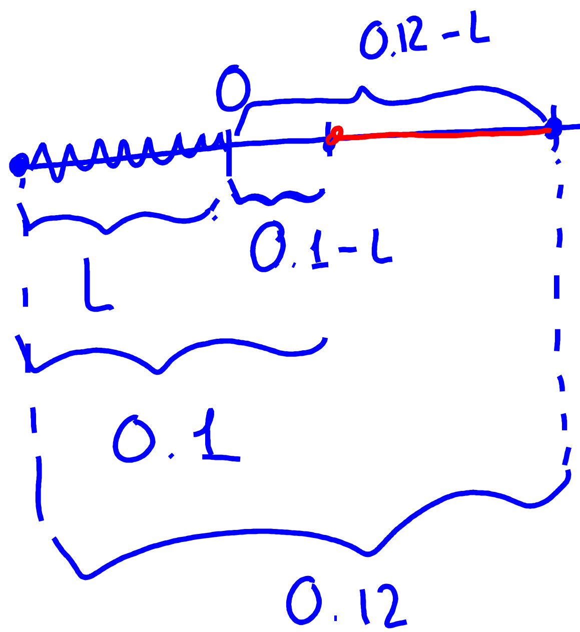
E.g. Suppose that : $\boxed{6\text{ J}}$ of work is required to stretch a spring from 10 cm to 12 cm.

10 J of work is required to stretch a spring from 12 cm to 14 cm.

Question: Find the natural length L of the spring.



$$W = \int_{0.1-L}^{0.12-L} f(x)\cdot dx = \int_{0.1-L}^{0.12-L} kx\,dx$$

$$\boxed{\int_{0.1-L}^{0.12-L} kx\,dx = 6}$$

$$\boxed{\int_{0.12-L}^{0.14-L} kx\,dx = 10}$$

$$\boxed{\int_{0.1-L}^{0.12-L} kx\,dx} = 6$$

$$k\int_{0.1-L}^{0.12-L} x\,dx = 6 \;; \quad k\cdot\frac{x^2}{2}\Bigg|_{0.1-L}^{0.12-L} = 6$$

$$\boxed{\frac{k}{2}\left[\boxed{(0.12-L)^2-(0.1-L)^2}\right] = 6}$$

$$\int_{0.12-L}^{0.14-L} kx\,dx = 10 \;; \quad \frac{k}{2}x^2\Bigg|_{0.12-L}^{0.14-L} = 10$$

$$\boxed{\frac{k}{2}\left[(0.14-L)^2-(0.12-L)^2\right] = 10}$$

Goal: Solve for L

$$\frac{k}{2}\left[(0.12-L)^2-(0.1-L)^2\right]=6$$

$$\frac{k}{2}\left[(0.12)^2 \boxed{-0.24L} + \cancel{L^2} - (0.1)^2 + \boxed{0.2L} - \cancel{L^2}\right]=6$$

$$\frac{k}{2}\left[0.0144-0.04L-0.01\right]=6$$

$$\boxed{\frac{k}{2}\left[0.0044-0.04L\right]=6}$$

$$\boxed{k(0.0044-0.04L)=12}$$

$$k=\frac{12}{0.0044-0.04L}$$

$$\frac{k}{2}\left[(0.14-L)^2-(0.12-L)^2\right]=10$$

$$\frac{k}{2}\left[(0.14)^2-0.28L+\cancel{L^2}-0.0144+0.24L-\cancel{L^2}\right]=10$$

$$\frac{k}{2}\left(0.0196-0.04L-0.0144\right]=10$$

$$\frac{k}{2}\left[0.0052-0.04L\right]=10$$

$$\boxed{k\left[0.0052-0.04L\right]=20}$$

$$\frac{12}{\boxed{0.0044 - 0.04L}} \cdot \left[0.0052 - 0.04L\right] = 20$$

$$12 \cdot \left[0.0052 - 0.04L\right] = 20\left[0.0044 - 0.04L\right]$$

$$\ldots \quad L = ?$$

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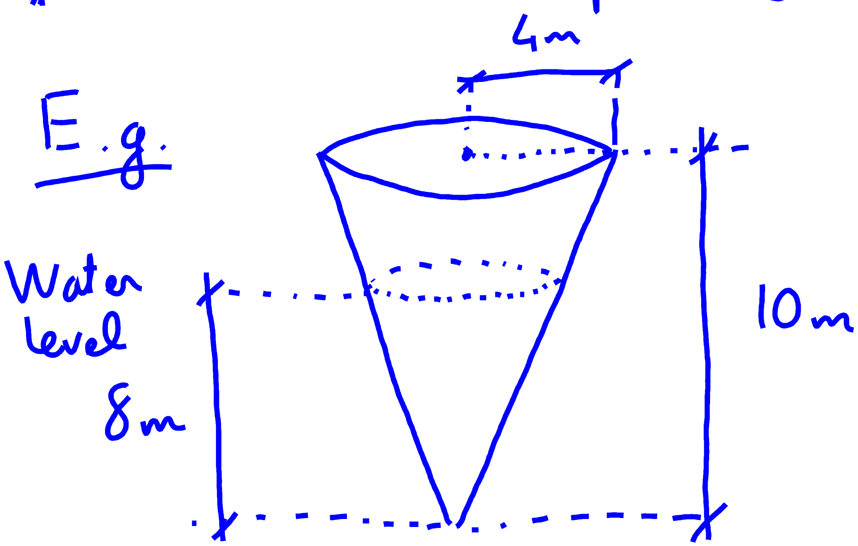
* Work done in pumping water (liquid) out of a tank.

E.g.

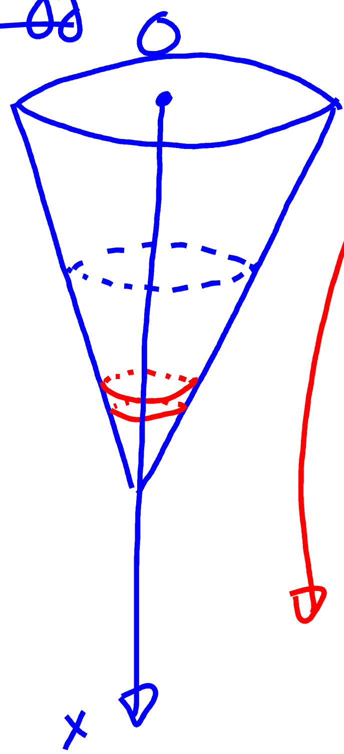


Tank: inverted cone as in the picture.

Density of water: $1000\ kg/m^3$

Find the work required to empty tank by pumping water over the top of the tank.

Strategy



* Find the work done in pumping a thin slice of water out of the tank.

* Integrate the above to find the total work in pumping the entire body of water out of the tank

work = (force) × (distance)

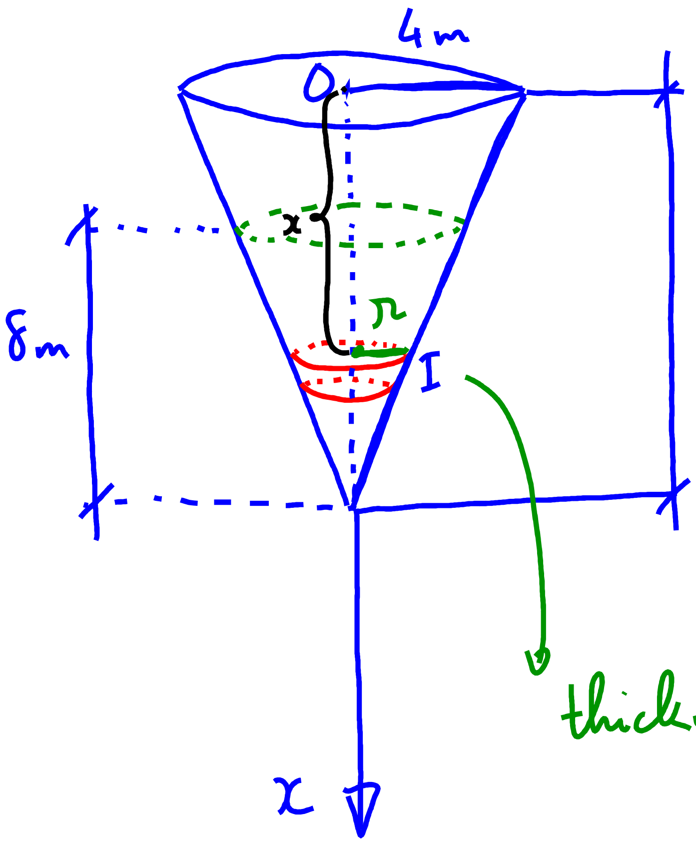
= (gravity on slice) × (distance)

= (mg) × distance

= (mass = density × volume)

= $\underline{(\text{density})} \times (\boxed{\text{volume}}) \times g \times \underline{(\text{distance})}$

It comes down to finding the formula of that thin slice of water.

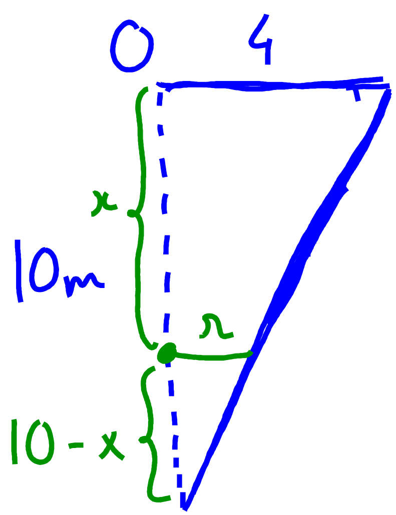


Volume of a generic slice:

$$V_{slice} = (\text{base area}) \cdot (\text{thickness})$$

$$= (\pi \cdot r^2) \cdot (dx)$$

thickness of slice $= dx$



By similar triangles:

$$\frac{r}{10-x} = \frac{4}{10} = \frac{2}{5}$$

$$r = \frac{2}{5}(10-x)$$

$$V_{slice} = \pi \cdot \left[\frac{2}{5}(10-x)\right]^2 \cdot dx$$

$$= \frac{4\pi}{25}(10-x)^2 \, dx$$

$$m_{slice} = (\text{density}) \cdot (V_{slice})$$

$$= 1000 \cdot \frac{4\pi}{25}(10-x)^2 \, dx$$

$$= 160\pi(10-x)^2 \, dx$$

$$\text{gravity}_{\text{slice}} = (m_{\text{slice}}) \cdot (g)$$

$$= \underbrace{(9.8) \cdot 160\pi (10-x)^2\,dx}_{\text{force acting on this slice = force required to lift this slice.}}$$

$$\text{work}_{\text{slice}} = (\text{force}) \cdot (\text{distance})$$

$$= \left[(9.8) \cdot 160\pi (10-x)^2\,dx \right] \cdot x$$

$$= (9.8) \cdot 160\pi (10-x)^2 x\,dx$$

Total work to pump the entire body of water:

$$W = \int_2^{10} (9.8) \cdot 160\pi (10-x)^2 x\,dx = (9.8) \cdot 160\pi \cdot \int_2^{10} (10-x)^2 x\,dx$$