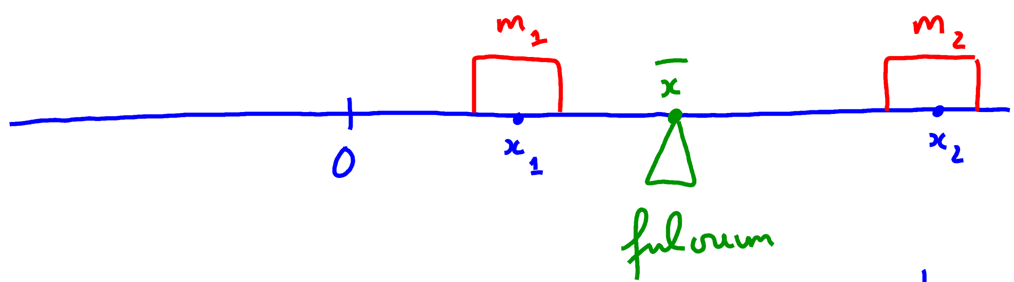
2.6. Moments and Centers of Mass

* 2 point masses m_1, m_2 located on the number line at x-coord x_1 and x_2.



The center of mass $\bar{x}$ is the point on the number line where we should place the fulcrum to make the system balance.

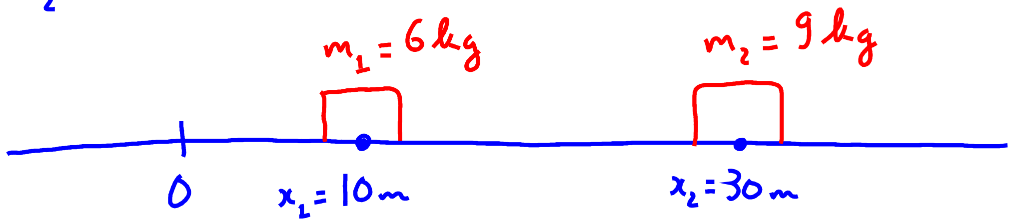
The formula for $\bar{x}$ is the "weighted average" of x_1 and x_2.

$$\boxed{\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}}$$

The quantity $m_1 x_1 + m_2 x_2$ is called the first moment of the system with respect to the origin.

$m_1 + m_2 = m$ is the total mass of the system.

E.g.



$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{6 \cdot 10 + 9 \cdot 30}{6 + 9} = 22\ (m)$$

In general, if we have n masses, $m_1, m_2, \ldots, m_n$ located at $x_1, x_2, \ldots, x_n$ then the center of mass $\bar{x}$ of the system is:

$$\boxed{\bar{x} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}}$$

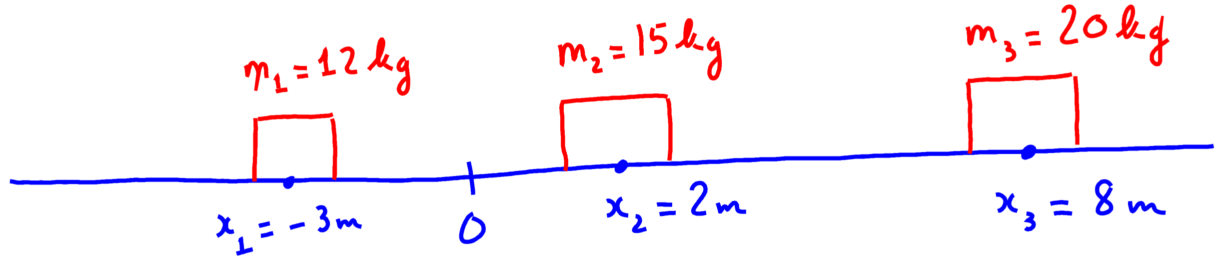
moment

total mass

The quantity
$$M = \sum_{i=1}^{n} m_i x_i$$
is called the <u>moment of the system</u>

The total mass of the system is $m = \sum_{i=1}^{n} m_i$.

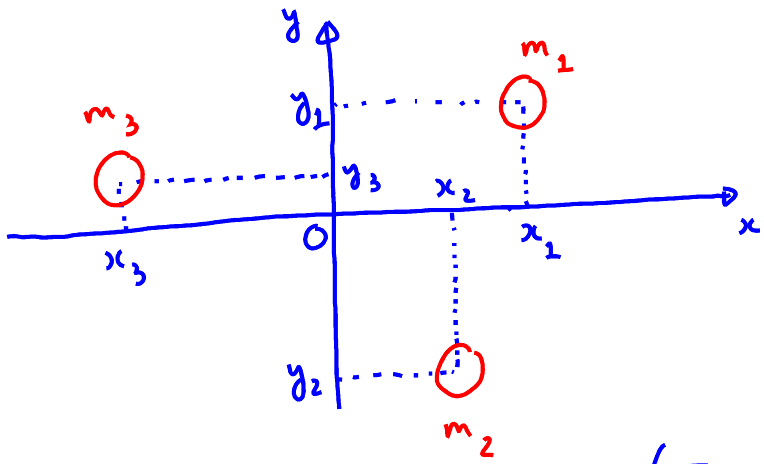
E.g.



$$\bar{x} = ?$$

$$\bar{x} = \frac{\sum_{i=1}^{3} m_i x_i}{\sum_{i=1}^{3} m_i} = \frac{12 \cdot (-3) + 15 \cdot 2 + 20 \cdot 8}{12 + 15 + 20} = \frac{154}{47} \approx 3.277 \ (m).$$

Now, consider a system of n masses located at n points (x_1, y_1); (x_2, y_2); $\dots$; (x_n, y_n) in the plane.



Picture for $n = 3$

The center of mass is the point $(\bar{x}, \bar{y})$ where the coordinates $\bar{x}, \bar{y}$ are given by

$$\boxed{\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}} \quad ; \quad \boxed{\bar{y} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i}}$$

$$\sum_{i=1}^{n} m_i = m : \text{ total mass.}$$

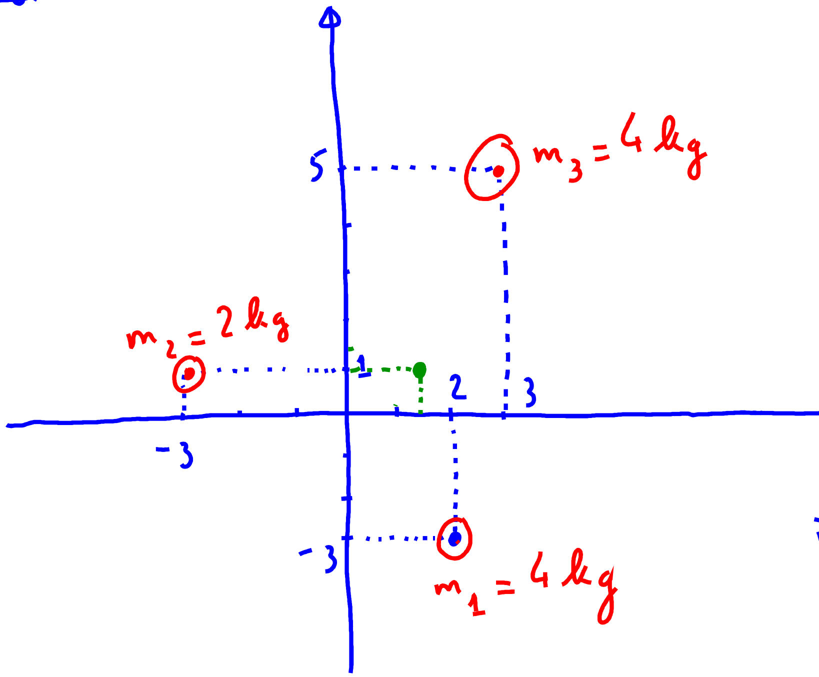
$$\sum_{i=1}^{n} m_i x_i = M_y$$: y-moment (or moment of system about y-axis)
it measures the tendency of the system to rotate about the y-axis.

$$\sum_{i=1}^{n} m_i y_i = M_x$$: x-moment (or moment of system about x-axis)
it measures the tendency of the system to rotate about the x-axis.

To sum up,

$$\boxed{\bar{x} = \frac{M_y}{m} \quad ; \quad \bar{y} = \frac{M_x}{m}}$$

E.g.



$(\bar{x}, \bar{y}) = ?$

$$\bar{x} = \frac{M_y}{m} = \frac{\sum_{i=1}^{3} m_i x_i}{m}$$

$$= \frac{4\cdot(2) + 2\cdot(-3) + 4\cdot(3)}{4+2+4} = \frac{7}{5}$$

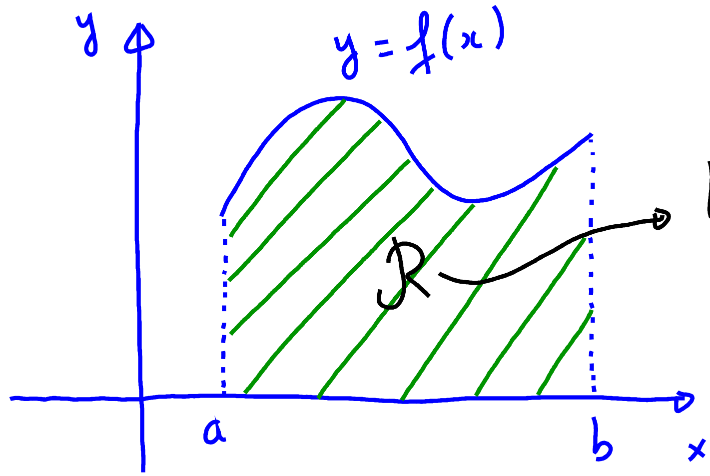
$$\boxed{\bar{x} = \frac{7}{5}}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\sum_{i=1}^{3} m_i y_i}{m}$$

$$\bar{y} = \frac{4\cdot(-3) + 2\cdot(1) + 4\cdot(5)}{4+2+4} = \boxed{1}$$

So, the center of mass is at the point $\boxed{\left(\frac{7}{5}, 1\right)}$. (green point in picture)

Center of mass of thin plates. (thin plate = lamina)



Lamina with uniform density ρ.
(Region in plane bounded by $y = f(x)$; $x = d$; $x = b$)

How do we find the center of mass $(\bar{x}, \bar{y})$?

(Btw, the center of mass of a lamina is also called the centroid of lamina)

Moment of R about y-axis: $\boxed{M_y = \rho \int_a^b x f(x)\,dx}$

Moment of R about x-axis: $\boxed{M_x = \rho \int_a^b \frac{1}{2}[f(x)]^2\,dx}$

Mass of R (from previous section): $\boxed{m = \rho \int_d^b f(x)\,dx}$

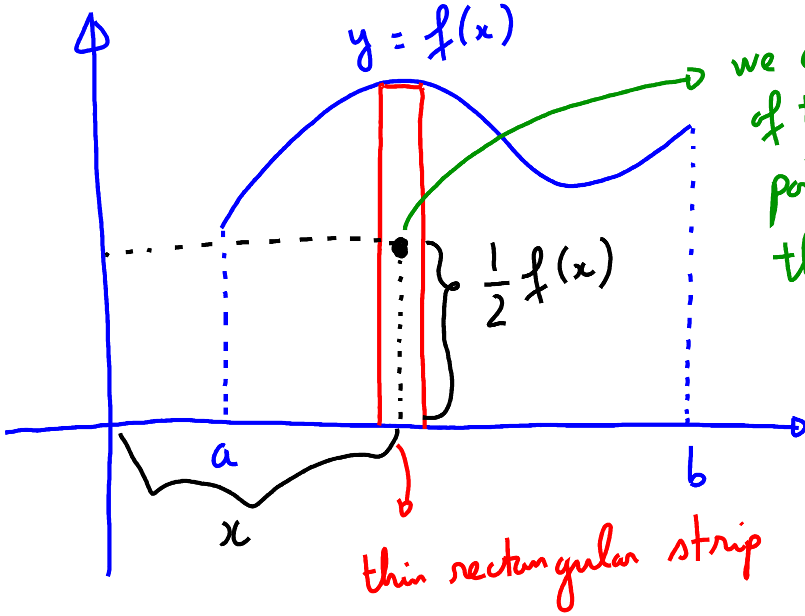
Centroid $(\bar{x}, \bar{y})$ of the region:

$$\bar{x} = \frac{M_y}{m} = \frac{\cancel{\rho}\int_a^b x\, f(x)\, dx}{\cancel{\rho}\int_a^b f(x)\, dx} = \boxed{\frac{\int_a^b x\, f(x)\, dx}{\int_a^b f(x)\, dx}}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\cancel{\rho}\int_a^b \frac{1}{2}[f(x)]^2\, dx}{\cancel{\rho}\int_a^b f(x)\, dx} = \boxed{\frac{\int_a^b \frac{1}{2}[f(x)]^2\, dx}{\int_a^b f(x)\, dx}}$$

Where does the formula for M_x and M_y come from?

Divide R into "thin" rectangular strips, find moments for the strips and add them up.



we can assume all the masses of the strip concentrate at this point at the center of the strip.
the coordinates of that point equal
$(x, \frac{1}{2}f(x))$

$$\text{Mass of strip} = \underbrace{\rho\ \underbrace{\overbrace{f(x)}^{\text{height}}\ \overbrace{dx}^{\text{width}}}_{\text{area}}}_{(\text{density})\times(\text{area})}$$

$$y\text{-moment } (M_y) \text{ of strip} = \underbrace{(\rho f(x)dx)}_{\text{mass}} \cdot \underbrace{x}_{x\text{-coord.}}$$

$$= \rho\, x\, f(x)\, dx$$

$$x\text{-moment } (M_x) \text{ of strip} = \underbrace{(\rho f(x)dx)}_{\text{mass}} \cdot \underbrace{\tfrac{1}{2}f(x)}_{y\text{-coord}}$$

$$= \rho \cdot \frac{1}{2}[f(x)]^2\, dx$$

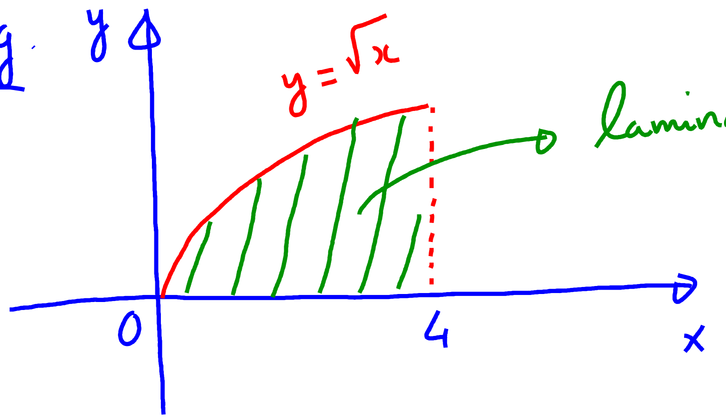
So, total y - moment = Sum of y - moments of strips

$$M_y(\text{system}) = \rho \int_a^b x\, f(x)\, dx.$$

total x - moment = Sum of x - moments of strips

$$M_x(\text{system}) = \rho \int_a^b \frac{1}{2}\left[f(x)\right]^2 dx.$$

E.g.



lamina with uniform density ρ.

Find the centroid $(\bar{x}, \bar{y})$

$$M_y = \rho \int_0^4 x\, f(x)\,dx = \rho \int_0^4 x \cdot \sqrt{x}\, dx = \rho \int_0^4 x^{3/2}\, dx$$

$$= \rho \cdot \frac{2x^{5/2}}{5} \Big|_0^4 = \rho \cdot \frac{2}{5} \cdot (4)^{\frac{5}{2}} = \frac{64}{5}\rho .$$

$$M_x = \rho \int_0^4 \frac{1}{2}\left[f(x)\right]^2 dx = \frac{\rho}{2} \int_0^4 (\sqrt{x})^2 dx = \frac{\rho}{2} \cdot \int_0^4 x\,dx$$

$$= \frac{\rho}{2} \cdot \frac{x^2}{2} \Big|_0^4 = 4\rho .$$

$$m = \rho \int_0^4 f(x)\,dx = \rho \cdot \int_0^4 \sqrt{x}\,dx = \rho \cdot \int_0^4 x^{\frac{1}{2}} dx = \rho \cdot \frac{2x^{\frac{3}{2}}}{3} \Big|_0^4$$

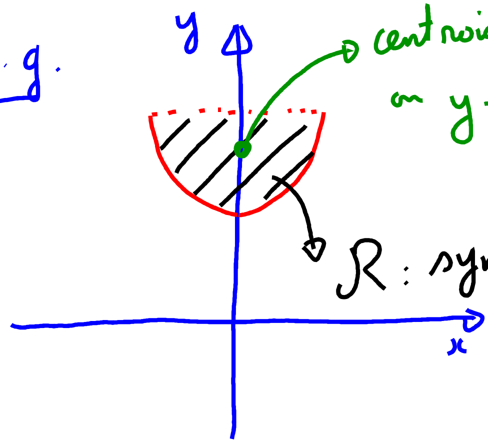
$$= \rho \cdot \frac{2}{3} \cdot (4)^{3/2} = \frac{16\rho}{3} .$$

$$\bar{x} = \frac{M_y}{m} = \frac{\frac{64}{5}\cancel{\rho}}{\frac{16}{3}\cancel{\rho}} = \boxed{\frac{12}{5}} . \quad \bar{y} = \frac{M_x}{m} = \frac{4\cancel{\rho}}{\frac{16}{3}\cancel{\rho}} = \boxed{\frac{3}{4}} . \quad \boxed{\text{Centroid}: \left(\frac{12}{5}, \frac{3}{4}\right)}$$

The Symmetry Principle

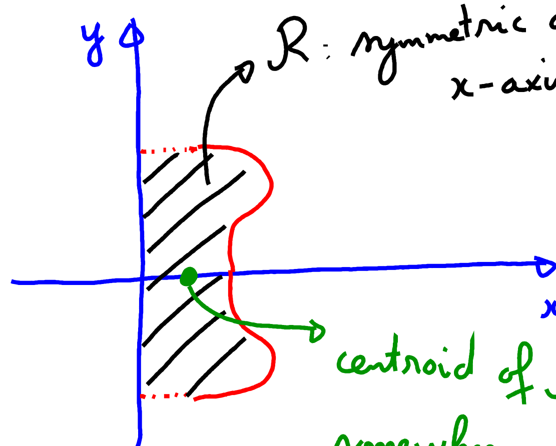
If a region $\mathcal{R}$ is symmetric about a line L, then the centroid of $\mathcal{R}$ lies on the line L.

E.g.



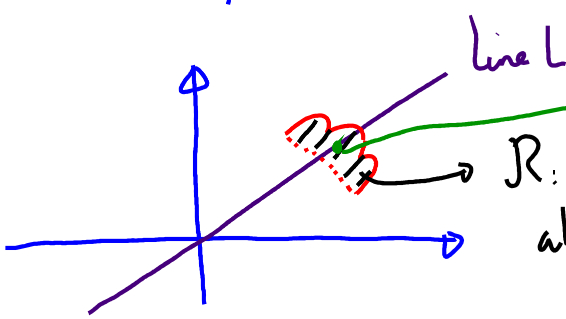
centroid of $\mathcal{R}$ is somewhere on y-axis

$\mathcal{R}$: symmetric about y-axis $(x=0)$



$\mathcal{R}$: symmetric about x-axis $(y=0)$

centroid of $\mathcal{R}$ is somewhere on the x-axis.



$\mathcal{R}$: symmetric about the line L

centroid of $\mathcal{R}$ lies somewhere on L.