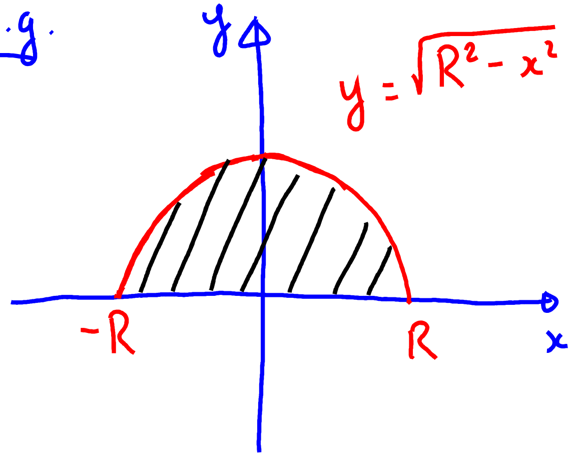
E.g.



$\mathcal{R}$: region bounded by the semicircle $y = \sqrt{R^2 - x^2}$ and the x-axis with uniform density ρ.

Find the centroid $(\bar{x}, \bar{y})$ of $\mathcal{R}$.

By the Symmetry Principle, since $\mathcal{R}$ is symmetric about the y-axis, the centroid of $\mathcal{R}$ must lie on the y-axis.

Therefore, $\boxed{\bar{x} = 0}$. Now, we only need to find $\bar{y}$.

$$\bar{y} = \frac{M_x}{m}. \qquad M_x = \rho \cdot \int_{-R}^{R} \frac{1}{2} [f(x)]^2 \, dx$$

$$\text{So, } M_y = \frac{\rho}{2} \int_{-R}^{R} \left(\sqrt{R^2 - x^2}\right)^2 dx = \frac{\rho}{2} \int_{-R}^{R} (R^2 - x^2)\, dx$$

$$= \frac{\rho}{2} \cdot \left(R^2 x - \frac{x^3}{3}\right)\Bigg|_{-R}^{R} = \frac{\rho}{2} \cdot \left[\left(R^3 - \frac{R^3}{3}\right) - \left(-R^3 + \frac{R^3}{3}\right)\right]$$

$$= \frac{\rho}{2} \cdot \left(\frac{4R^3}{3}\right) = \frac{2R^3\rho}{3}.$$

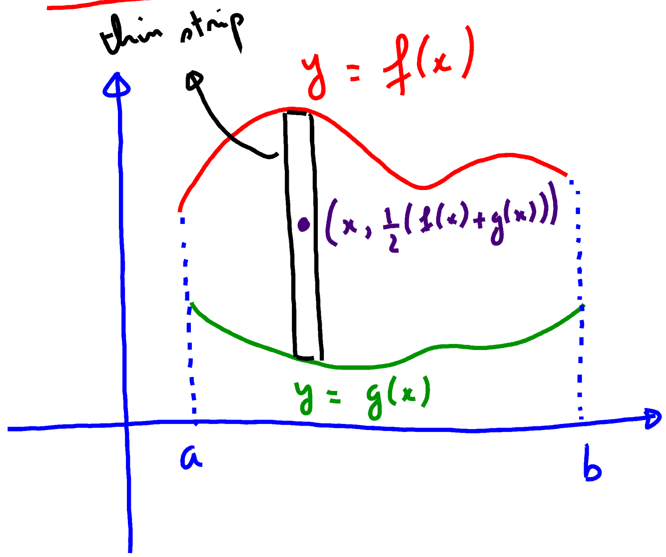
$$m = \rho \cdot (\text{area of region}) = \rho \cdot (\text{area of semicircle})$$

$$= \rho \cdot \left(\frac{\pi R^2}{2}\right) = \frac{\pi R^2 \rho}{2}.$$

$$\bar{y} = \frac{M_x}{m} = \frac{2R^3\rho/3}{\pi R^2 \rho/2} = \frac{4R}{3\pi}. \qquad \boxed{\text{Centroid: } \left(0, \frac{4R}{3\pi}\right)}$$

Centroid of region bounded by 2 curves.

Same reasoning as before, we got:



y - moment of region:

$$\boxed{M_y = \rho \int_a^b x \left[f(x) - g(x) \right] dx}$$

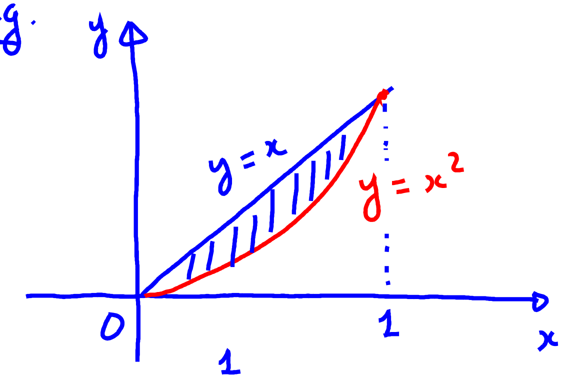
x - moment of region

$$\boxed{M_x = \rho \int_a^b \frac{1}{2} \left[(f(x))^2 - (g(x))^2 \right] dx}$$

mass of region:

$$\boxed{m = \rho \int_d^b \left[f(x) - g(x) \right] dx.}$$

E.g.



Centroid $(\bar{x}, \bar{y}) = ?$

$$M_{ass} = \rho \int_0^1 (x - x^2)\, dx = \rho \cdot \left(\frac{x^2}{2} - \frac{x^3}{3} \right) \Bigg|_0^1 = \frac{1}{2} - \frac{1}{3} = \boxed{\frac{1}{6}} \cdot \rho$$

$$M_y = \rho \int_0^1 x \cdot (x - x^2)\, dx = \rho \cdot \int_0^1 (x^2 - x^3)\, dx = \rho \cdot \left(\frac{x^3}{3} - \frac{x^4}{4} \right) \Bigg|_0^1 = \boxed{\frac{1}{12} \rho}$$

$$M_x = \rho \int_0^1 \frac{1}{2} \cdot \left[x^2 - x^4 \right] dx = \frac{\rho}{2} \cdot \left(\frac{x^3}{3} - \frac{x^5}{5} \right) \Bigg|_0^1 = \frac{\rho}{2} \cdot \frac{2}{15} = \boxed{\frac{1}{15} \rho}$$

$$\bar{x} = \frac{M_y}{m} = \frac{\rho/12}{\rho/6} = \frac{1}{2}. \qquad \bar{y} = \frac{M_x}{m} = \frac{\rho/15}{\rho/6} = \frac{2}{5}.$$

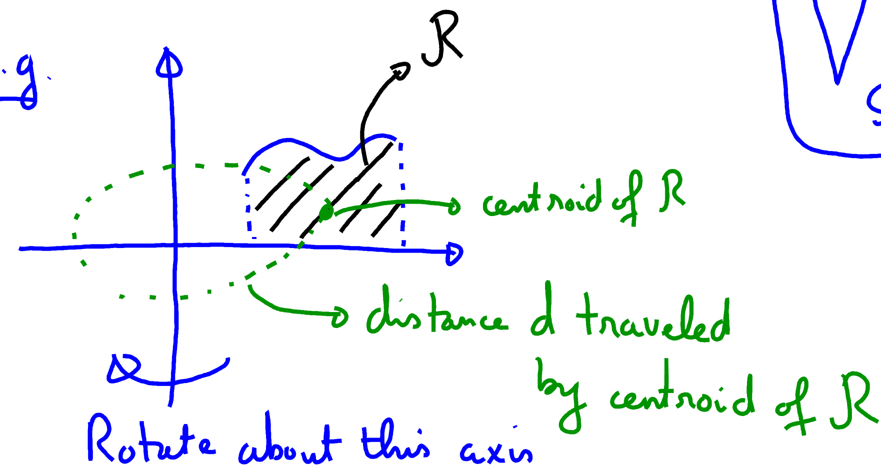
$$\boxed{\text{Centroid: } \left(\frac{1}{2}, \frac{2}{5} \right)}$$

Theorem of Pappus for Volume.

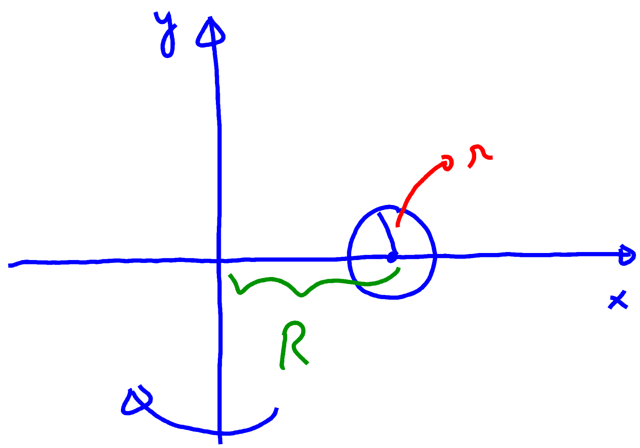
If a solid S is obtained by rotating a region $\mathcal{R}$ about an axis, then the volume of S is equal to the product of the area of $\mathcal{R}$ and the distance traveled by the centroid of $\mathcal{R}$.

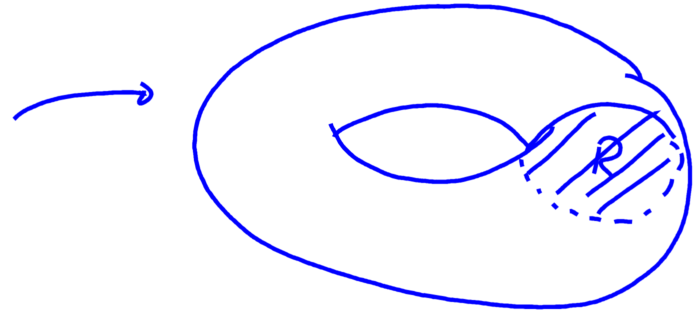
$$\boxed{V_{\text{Solid } S} = (\text{Area of } \mathcal{R}) \cdot (d)}$$

E.g.



E.g. Volume of torus





Solid: dognut shape.

$V = (\text{area of } R) \cdot (\text{distance traveled by centroid})$

$= \underbrace{(\pi r^2)}_{\text{area of circle}} \cdot (2\pi R) = \boxed{2\pi^2 r^2 R}$ → Volume of dognut.

$(2\pi R)$ → distance traveled by center after 1 full revolution.