3.2. Trigonometric Integrals

Review of Integration by Parts

Formula : $\displaystyle\int u\,dv = uv - \int v\,du$

How do we choose u and dv ?

Often, LIATE works

L : Log function
I : Inverse trig functions
A : Algebraic functions
T : Trig function
E : Exponential function

whichever type of function appears first in the list should be u

E.g. $\int \underbrace{x^3}_{\text{algebraic}} \cdot \underbrace{\ln x}_{\text{Log}} \, dx$ L comes before A in LIATE

So, $\begin{cases} u = \ln x \\ dv = x^3 dx \end{cases} \Rightarrow \begin{cases} du = \frac{1}{x} dx \\ v = \int x^3 dx = \frac{x^4}{4} \end{cases}$

$$\int \underbrace{(\ln x)}_{u} \cdot \underbrace{x^3 dx}_{dv} = \underbrace{(\ln x)}_{u} \cdot \underbrace{\left(\frac{x^4}{4}\right)}_{v} - \int \underbrace{\frac{x^4}{4}}_{v} \cdot \underbrace{\frac{1}{x} dx}_{du}$$

$$= \frac{x^4 \cdot \ln x}{4} - \int \frac{x^3}{4} dx = \boxed{\frac{x^4 \cdot \ln x}{4} - \frac{x^4}{16} + C}$$

E.g. $\int \underbrace{\tan^{-1} x}_{\text{Inverse Trig}} \, dx$ → $\underbrace{1}_{\text{algebraic}} \cdot dx$ LIATE

$\begin{cases} u = \tan^{-1} x \\ dv = dx \end{cases} \Rightarrow \begin{cases} du = \frac{1}{1+x^2} dx \\ v = x \end{cases}$

$$\int \tan^{-1} x \, dx = x \tan^{-1} x - \boxed{\frac{1}{2} \int \frac{2 \cdot x \, dx}{\underbrace{1+x^2}_{w}}}$$

dw

→ $w = 1 + x^2$

$dw = 2x dx$

$$= x \tan^{-1} x - \frac{1}{2} \int \frac{dw}{w} = x \tan^{-1} x - \frac{1}{2} \ln |1 + x^2| + C$$

Today: Integration of Trig Functions

Forms

Preliminary form: $\int \sin^k x \cos x\, dx$; $\int \cos^k x \sin x\, dx$

$\int \sin^m x \cos^n x\, dx$
- power of sine (m) is odd
- power of cosine (n) is odd
- both m and n are even

$\int \tan^m x \sec^n x\, dx$
- power of secant (n) is even
- power of tangent (m) is odd
- other cases

$\int \sin(mx)\sin(nx)\, dx$; $\int \cos(mx)\cos(nx)\, dx$

$\int \sin(mx)\cos(nx)\, dx$ ⟵ product-to-sum trig identities.

Preliminary form: $\underbrace{\int \sin^k x \cos x \, dx}_{u = \sin x}$ or $\underbrace{\int \cos^k x \sin x \, dx}_{u = \cos x}$

Strategy: u - sub

Key formula: $\frac{d}{dx}(\sin x) = \cos x$; $\frac{d}{dx}(\cos x) = -\sin x$

E.g. $-\int \cos^4 x (-\sin x \, dx)$; $u = \cos x$; $du = -\sin x \, dx$

$$= -\int u^4 du = -\frac{u^5}{5} + C = -\frac{(\cos x)^5}{5} + C = \boxed{\frac{-\cos^5 x}{5} + C}$$

Form: $\int \sin^n x \cos^m x \, dx$

$\cos^6 x = (\cos^2 x)^3$

$(1-u^2)$

Case 1: m, the power of cosine, is odd

E.g. $\int \sin^2 x \cos^{\boxed{3}} x \, dx = \int \underbrace{\sin^2 x}_{u^2} \underbrace{(\cos^2 x)}_{(1-u^2)} \underbrace{(\cos x \, dx)}_{du}$

let $u = \sin x$, $du = \cos x \, dx$

$\sin^2 x + \cos^2 x = 1 \iff \cos^2 x = 1 - \sin^2 x$

$$\int u^2 \cdot (1-u^2) \, du = \int (u^2 - u^4) \, du = \frac{u^3}{3} - \frac{u^5}{5} + C$$

$$= \boxed{\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C}$$

$\int \sin^{2017} x \cos^{\boxed{5}} x \, dx$

$\int \sin^{2017} x \cos^4 x \, \underline{\cos x \, dx}$

$u = \sin x$

$\int u^{2017} (1-u^2)^2 \, du$

Strategy (power of cosine is odd)

1. Save a cosine factor
2. Use the identity: $\boxed{\cos^2 x = 1 - \sin^2 x}$ to express the rest of the integral in terms of $\sin x$
3. Let $\boxed{u = \sin x}$.

Case 2: $\int \sin^n x \cos^m x \, dx$, where n, the power of sine is odd.

E.g. $\int \sin^3 x \cos^4 x \, dx$

let $u = \cos x$
$du = -\sin x \, dx$

$$\int \underbrace{\sin^2 x}_{1-u^2} \underbrace{\cos^4 x}_{u^4} \underbrace{(\sin x \, dx)}_{(-du)}$$

$$\int (1-u^2) u^4 (-du) = -\int (1-u^2) u^4 \, du$$

$$= -\int (u^4 - u^6) \, du = -\frac{u^5}{5} + \frac{u^7}{7} + C = \boxed{-\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C}$$

Strategy: Power of sine is odd

1. Save one factor of sine
2. Use the identity $\boxed{\sin^2 x = 1 - \cos^2 x}$ to express the rest of the integral in terms of cosine.
3. let $u = \cos x$

Case 3: $\int \sin^n x \cos^m x \, dx$ Both m and n are even

Strategy: Use power reduction formula.

Key formula:

$$\left.\begin{aligned} \sin^2 x &= \frac{1}{2}\left(1 - \cos(2x)\right) \\ \cos^2(x) &= \frac{1}{2}\left(1 + \cos(2x)\right) \end{aligned}\right\} \text{Power reduction}$$

Sometimes, double angle formula for sine could be useful:

$$\sin(2x) = 2 \sin x \cos x$$

E.g.

$$\int \sin^2 x \, dx = \int \frac{1}{2}\left(1 - \cos(2x)\right) dx = \frac{1}{2} \int \left(1 - \cos(2x)\right) dx$$

$$= \frac{1}{2}\left(x - \frac{1}{2}\sin(2x)\right) + C.$$

$$\int \cos^2 x \, dx = \int \frac{1}{2}\left(1 + \cos(2x)\right) dx = \frac{1}{2}\left(x + \frac{1}{2}\sin(2x)\right) + C$$

$$\int \sin^4 x \, dx = \int \left(\sin^2 x\right)^2 dx = \int \left[\frac{1}{2}\left(1 - \cos(2x)\right)\right]^2 dx$$

$$= \frac{1}{4} \int \left(1 - \cos(2x)\right)^2 dx = \frac{1}{4} \int \left(1 - 2\cos(2x) + \underbrace{\cos^2(2x)}_{\boxed{\frac{1}{2}\left(1 + \cos(4x)\right)}}\right) dx$$

$$\boxed{(A - B)^2 = A^2 - 2AB + B^2}$$

$$= \frac{1}{4} \int \left(1 - 2\cos(2x) + \frac{1}{2}\left(1 + \cos(4x)\right)\right) dx$$

$$\frac{1}{4}\int\left(1-2\cos(2x)+\frac{1}{2}+\frac{1}{2}\cos(4x)\right)dx$$

$$=\frac{1}{4}\int\left(\frac{3}{2}-2\cos(2x)+\frac{1}{2}\boxed{\cos(4x)}\right)dx$$

$$=\frac{1}{4}\left[\frac{3}{2}x-\sin(2x)+\frac{1}{8}\sin(4x)\right]+C.$$

Form: $\int \tan^n x \sec^m x \, dx$

Key formulas to keep in mind:

$$\frac{d}{dx}(\tan x)=\sec^2 x \quad ; \quad \frac{d}{dx}(\sec x)=\sec x\tan x$$

$$\boxed{\int \sec^2 x\,dx=\tan x+C} \; ; \; \boxed{\int \sec x\tan x\,dx=\sec x+C}$$

$$\boxed{\int \tan x\,dx=\ln|\sec x|+C} \; ; \; \boxed{\int \sec x\,dx=\ln|\sec x+\tan x|+C}$$

$$-\int\frac{-\sin x}{\cos x}dx \; ; \quad u=\cos x,\ du=-\sin x\,dx$$

$$=-\int\frac{du}{u}=-\ln|u|+C=-\ln|\cos x|+C=\ln|\cos x|^{-1}+C$$

$$=\ln|\sec x|+C$$

Case 1: $\int \tan^m x \sec^n x\, dx$ where n, the power of secant is even.

E.g. $\int \tan^6 x \sec^4 x\, dx$

let $\boxed{u = \tan x}$

$du = \sec^2 x\, dx$

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\boxed{\tan^2 x + 1 = \sec^2 x}$$

$$= \int \underbrace{\tan^6 x}_{u^6} \cdot \underbrace{\sec^2 x}_{(1+u^2)} \cdot \underbrace{\sec^2 x\, dx}_{du}$$

$$= \int u^6 \cdot (1 + u^2)\, du$$

$$= \int (u^6 + u^8)\, du = \frac{u^7}{7} + \frac{u^9}{9} + C$$

$$= \boxed{\frac{\tan^7 x}{7} + \frac{\tan^9 x}{9} + C}$$

Strategy: (Power of secant is even)

1. Save a factor of $\sec^2 x$
2. Use the identity $\sec^2 x = 1 + \tan^2 x$ to express the rest of the integral in terms of tangent
3. let $u = \tan x$

Case 2: $\int \tan^m x \sec^n x \, dx$, where m, the power of tangent is odd

E.g $\int \tan^5 x \sec^7 x \, dx$

let $u = \sec x$

$$= \int \underbrace{\tan^4 x}_{(u^2-1)^2} \underbrace{\sec^6 x}_{u^6} \underbrace{(\sec x \tan x \, dx)}_{du}$$

$\boxed{\tan^2 x = \sec^2 x - 1}$
$= u^2 - 1$

$\tan^4 x = (\tan^2 x)^2$
$= (u^2 - 1)^2$

$$= \int (u^2 - 1)^2 \cdot u^6 \, du$$

$$= \int (u^4 - 2u^2 + 1) u^6 \, du = \int (u^{10} - 2u^8 + u^6) \, du$$

$$= \frac{u^{11}}{11} - 2\frac{u^9}{9} + \frac{u^7}{7} + C = \boxed{\frac{\sec^{11} x}{11} - 2 \cdot \frac{\sec^9 x}{9} + \frac{\sec^7 x}{7} + C}$$

Strategy: (the power of tangent is odd)

1. Save a factor of $\sec x \tan x$
2. Use $\boxed{\tan^2 x = \sec^2 x - 1}$ to express the rest of integral in terms of $\sec x$
3. let $u = \sec x$

Case 3: Neither 1 nor 2

This could require a variety of techniques: by parts, trig identities, u-sub, etc.

E.g. $\int \sec^3 x \, dx$

$$\begin{cases} u = \sec x \\ dv = \sec^2 x \, dx \end{cases} \qquad \begin{cases} du = \sec x \tan x \, dx \\ v = \int \sec^2 x \, dx = \tan x \end{cases}$$

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec x \tan^2 x \, dx$$

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx$$

$$\int \sec^3 x \, dx = \sec x \tan x \boxed{- \int \sec^3 x \, dx} + \boxed{\int \sec x \, dx}$$

$$2 \int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx$$

$$\boxed{\int \sec^3 x \, dx = \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x| + C\right)}$$

Form: $\int \sin(nx)\cos(mx)\,dx$; $\int \sin(nx)\sin(mx)\,dx$

$$\int \cos(nx)\cos(mx)\,dx.$$

$\longrightarrow$ Product-to-Sum Formula:

$$\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$$

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$

E.g. $$\int \cos\Big(\overset{A}{\boxed{6x}}\Big)\cos\Big(\overset{B}{\boxed{3x}}\Big)\,dx$$

$$\frac{1}{2}\int\left[\cos(3x) + \cos(9x)\right]dx = \frac{1}{2}\left[\frac{1}{3}\sin(3x) + \frac{1}{9}\sin(9x)\right] + C.$$