3.3. Trigonometric Substitutions

Goal: Find integrals where the integrands have radical expressions in them.

Strategy: Trig Substitution

Expression	Trig Sub	Identity	
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$; $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$	$\sqrt{a^2 - a^2\sin^2\theta}$ $= a\sqrt{1-\sin^2\theta}$ $= a\cos\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$ $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$	$\sqrt{a^2 + a^2\tan^2\theta}$ $= a\sqrt{1+\tan^2\theta}$ $= a\sec\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$ $0 \leq \theta < \frac{\pi}{2}$ on $\pi \leq \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$	$\sqrt{a^2\sec^2\theta - a^2}$ $= a\sqrt{\sec^2\theta - 1}$ $= a\tan\theta$

→ turn the integral into an integral of trig functions.

→ evaluate this new integral

→ Use geometry & trig identities to turn the result back to x

E.g. $\int \sqrt{9-x^2}\,\boxed{dx}$

Trig sub: let $\boxed{x = 3\sin\theta}$; $dx = \boxed{3\cos\theta\, d\theta}$

$\boxed{\left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)}$

$$\int \sqrt{9-(3\sin\theta)^2}\cdot 3\cos\theta\, d\theta$$

$$= \int \sqrt{9 - 9\sin^2\theta}\cdot 3\cos\theta\, d\theta$$

$$= \int \sqrt{9(1-\sin^2\theta)}\cdot 3\cos\theta\, d\theta = \int \sqrt{9\cos^2\theta}\cdot 3\cos\theta\, d\theta$$

$$= 9\int \cos^2\theta\, d\theta = 9\cdot\int \frac{1}{2}(1+\cos(2\theta))\, d\theta$$

$$= \frac{9}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + C = \frac{9\theta}{2} + \frac{9\sin(2\theta)}{4} + C$$

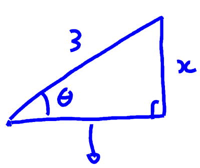
$$= \frac{9\theta}{2} + \frac{9\cdot 2\sin\theta\cos\theta}{4} + C = \underbrace{\frac{9\theta}{2} + \frac{9\sin\theta\cos\theta}{2} + C}$$

in terms of x

Recall: $x = 3\sin\theta$

$\frac{x}{3} = \sin\theta$

$\theta = \sin^{-1}\left(\frac{x}{3}\right)$

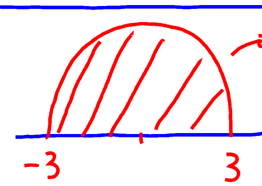


$\sqrt{9-x^2}$

$\cos\theta = \frac{\sqrt{9-x^2}}{3}$

$$\boxed{\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} + C}$$

Trick: for definite integral:



graph $\quad y = \boxed{\sqrt{9-x^2}}$

$y^2 = 9 - x^2$

$x^2 + y^2 = 9$

$$\int_{-3}^{3} \boxed{\sqrt{9-x^2}}\, dx = \text{Area} = \frac{\pi\cdot(3)^2}{2} = \boxed{\frac{9\pi}{2}}$$

Ex. $$\int \frac{\sqrt{9-x^2}}{x^2}\,dx \;;\quad x = 3\sin\theta \qquad dx = 3\cos\theta\, d\theta$$

$$\int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta}\cdot 3\cos\theta\, d\theta$$

$$\boxed{\begin{aligned}\cot^2\theta + 1 &= \csc^2\theta \\ \cot^2\theta &= \csc^2\theta - 1\end{aligned}}$$

$$= \int \frac{3\cos\theta}{9\sin^2\theta}\cdot 3\cos\theta\, d\theta = \int \frac{\cos^2\theta}{\sin^2\theta}\, d\theta$$

$$= \int \cot^2\theta\, d\theta = \int (\csc^2\theta - 1)\, d\theta$$

$$= -\cot\theta - \theta + C = \boxed{-\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C}$$

$x = 3\sin\theta \;;\; \sin\theta = \frac{x}{3}$

$\theta = \sin^{-1}\left(\frac{x}{3}\right) \qquad \cot\theta = \frac{\sqrt{9-x^2}}{x}$

$\sqrt{9-x^2}$

Ex.

$$\int \frac{dx}{\sqrt{4+x^2}} \quad ; \quad x = 2\tan\theta \quad ; \quad dx = 2\sec^2\theta \, d\theta$$

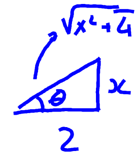
$$\int \frac{2\sec^2\theta \, d\theta}{\sqrt{4+4\tan^2\theta}} = \int \frac{2\sec^2\theta \, d\theta}{\sqrt{4(1+\tan^2\theta)}} = \int \frac{2\sec^2\theta \, d\theta}{\sqrt{4\sec^2\theta}}$$

$$= \int \frac{\cancel{2}\sec^2\theta \, d\theta}{\cancel{2}\sec\theta} = \int \sec\theta \, d\theta = \ln\left|\sec\theta + \tan\theta\right| + C$$

$$= \boxed{\ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + C}$$

$x = 2\tan\theta$

$\tan\theta = \frac{x}{2}$



$\sec\theta = \frac{\sqrt{x^2+4}}{2}$.

E.g. HW # 10

$$\int \frac{dx}{(x^2-36)^{3/2}} = \int \frac{dx}{\left(\sqrt{x^2-36}\right)^3}$$

$$x = 6\sec\theta \;; \quad dx = 6\sec\theta\tan\theta\, d\theta$$

$$\int \frac{6\sec\theta\tan\theta}{\left(\sqrt{36\sec^2\theta - 36}\right)^3}\,d\theta = 6\int \frac{\sec\theta\tan\theta}{\left[\sqrt{36\underbrace{(\sec^2\theta - 1)}_{\tan^2\theta}}\right]^3}\,d\theta$$

$$= 6 \cdot \int \frac{\sec\theta\,\cancel{\tan\theta}}{216\tan^{\cancel{3}\,2}\theta}\,d\theta = \frac{1}{36}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

$$\frac{1}{36}\int \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}\,d\theta = \frac{1}{36}\int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

let $u = \sin\theta$

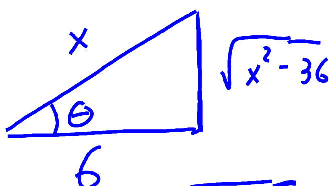
$du = \cos\theta\, d\theta$

$$= \frac{1}{36}\cdot\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$$= \frac{1}{36}\cdot\int \frac{du}{u^2} = \frac{1}{36}\cdot\int u^{-2}\,du = \frac{1}{36}\cdot\frac{u^{-1}}{-1} + C$$

$$= -\frac{1}{36u} + C = -\frac{1}{36\sin\theta} + C$$

$x = 6\sec\theta$; $\sec\theta = \frac{x}{6}$



$$\sin\theta = \frac{\sqrt{x^2-36}}{x}$$

$$= -\frac{1}{36\cdot\frac{\sqrt{x^2-36}}{x}} + C$$

$$= \boxed{-\frac{x}{36\sqrt{x^2-36}} + C}$$

HW #12

$$\int \frac{1}{\sqrt{\boxed{x^2+4x-12}}}\,dx$$

Completing the square:

$$\underbrace{x^2+4x}-12 = \underbrace{x^2+4x+4}\ \underbrace{-4-12}$$

$$= (x+2)^2-16$$

$$\int \frac{dx}{\sqrt{(x+2)^2-16}}$$

Trig sub: $\boxed{x+2 = 4\sec\theta}$

$dx = 4\sec\theta\tan\theta\,d\theta$

$$\int \frac{4\sec\theta\tan\theta}{\sqrt{(4\sec\theta)^2-16}}\,d\theta$$

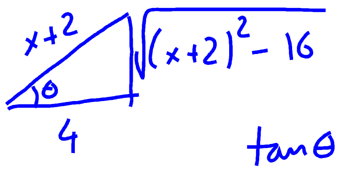
$$= 4\int \frac{\sec\theta\tan\theta}{4\sqrt{\sec^2\theta-1}} = \int \frac{\sec\theta\,\cancel{\tan\theta}}{4\,\cancel{\tan\theta}}\,d\theta$$

$$= \cancel{\frac{1}{4}}\int \sec\theta\,d\theta = \cancel{\frac{1}{4}}\ln\left|\sec\theta+\tan\theta\right|+C.$$

$$= \boxed{\cancel{\frac{1}{4}}\ln\left|\frac{x+2}{4}+\frac{\sqrt{(x+2)^2-16}}{4}\right|+C}$$

$$x+2 = 4\sec\theta$$

$$\sec\theta = \frac{x+2}{4}$$



$$\tan\theta = \frac{\sqrt{(x+2)^2-16}}{4}$$