3.4. Partial Fractions Decomposition.

Integrate $\underbrace{\text{Rational Functions}}$.

functions of the form $\dfrac{P(x)}{Q(x)}$; P, Q: polynomials.

We know: $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C.$ $\displaystyle\int \frac{du}{u} = \ln|u| + C$

$$\int \frac{1}{x+10}\,dx = \ln|x+10| + C$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\cdot \tan^{-1}\left(\frac{x}{a}\right) + C. \qquad \int \frac{du}{u^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

Today, Integrate $\displaystyle\int \frac{P(x)}{Q(x)}\,dx$ in general.

1st thing: if degree $P(x) \geqslant$ degree $Q(x)$, then we need to perform long division.

E.g. $\displaystyle\int \frac{x^2+3x+5}{x+1}\,dx$

degree top = 2 > degree bottom = 1.

$$\begin{array}{r|l} & \boxed{x+2} \longrightarrow \text{quotient} \\ \hline \boxed{x}+1 & \boxed{x^2}+3x+5 \\ & -(x^2+x) \\ \hline & 0 \quad \boxed{2x}+5 \\ & -(2x+2) \\ \hline & \boxed{3} \longrightarrow \text{Remainder.} \end{array}$$

$$\frac{x^2+3x+5}{x+1} = \underbrace{x+2}_{\text{quotient}} + \frac{\overset{\text{Remainder}}{\boxed{3}}}{x+1}$$

$$\int \boxed{\frac{x^2+3x+5}{x+1}}\,dx = \int \boxed{\left(x+2+\frac{3}{x+1}\right)}\,dx = \boxed{\frac{x^2}{2}+2x+3\ln|x+1| + C}$$

* Now assume degree $P(x) <$ degree $Q(x)$.

$$\int \frac{P(x)}{Q(x)}$$

$\longrightarrow$ Factor $Q(x)$

* When we factor $Q(x)$, there are a few scenarios.

(1) $Q(x)$ is a product of non repeated linear factors

E.g. $\displaystyle\int \frac{3x+2}{x^3 - x^2 - 2x}\,dx = \int \frac{3x+2}{x(x^2 - x - 2)}\,dx$

$$= \int \frac{3x+2}{x(x-2)(x+1)} \longrightarrow \text{product of non repeated linear factors}$$

completely factor $Q(x)$

It turns out that, we can find #'s A, B, C such that

$$\boxed{\frac{3x+2}{\boxed{x(x-2)(x+1)}}} = \boxed{\frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1}}$$

→ Partial Fractions Decomposition.

→ How do we find A, B, C ?

Multiply both sides by $x(x-2)(x+1)$

$$3x+2 = A(x-2)(x+1) + Bx(x+1) + Cx(x-2).$$

Goal: find #'s A, B, C s.t. the above equation is true for all x.

Method of equating coefficients

$$3x+2 = A(x^2-x-2) + B(x^2+x) + C(x^2-2x)$$

$$3x+2 = \underbrace{(A+B+C)}_{\text{coefficients}}x^2 + \underbrace{(-A+B-2C)}x + \underbrace{(-2A)}$$

$$\left.\begin{array}{ll}
\Rightarrow \text{there's no } x^2 \text{ on LHS} & \Rightarrow A+B+C=0 \\
\text{Since coefficient of } x \text{ on LHS} = 3 & \Rightarrow -A+B-2C=3 \\
\text{Constant term on LHS} = 2 & \Rightarrow -2A = 2
\end{array}\right.$$

→ $A = -1$

$$\Rightarrow \begin{cases} -1+B+C=0 \\ 1+B-2C=3 \end{cases} \Rightarrow \begin{cases} B+C=1 \\ B-2C=2 \end{cases} \rightarrow \begin{cases} B = 1-C \\ (1-C)-2C=2 \end{cases}$$

$$\Rightarrow \begin{cases} B = 1-C \\ 1-3C=2 \end{cases} \Rightarrow \begin{cases} B = \frac{4}{3} \\ C = -\frac{1}{3} \end{cases}$$

To sum up: $A = -1$; $B = \frac{4}{3}$; $C = -\frac{1}{3}$.

We have:

$$\int \frac{3x+2}{x(x-2)(x+1)}\,dx = \left(\int -\frac{1}{x} + \int \frac{4/3}{x-2} + \int \frac{-1/3}{x+1}\right) dx$$

$$= \boxed{-\ln|x| + \frac{4}{3}\ln|x-2| - \frac{1}{3}\ln|x+1| + C}$$

Here's another way to find A, B, C.

$$3x+2 = A(x-2)(x+1) + Bx(x+1) + Cx(x-2)$$

Goal: find A, B, C s.t. this equation is true for all x.

Method of strategic substitution:

choose $x=-1$: $\underbrace{-1}_{\text{LHS}} = \underbrace{3C}_{\text{RHS}} \Rightarrow \boxed{C = -1/3}$

choose $x=2$: $\underbrace{8}_{\text{LHS}} = \underbrace{6B}_{\text{RHS}} \Rightarrow \boxed{B = 4/3}$

choose $x=0$: $\underbrace{2}_{\text{LHS}} = -2A \Rightarrow \boxed{A = -1}$

Case 1 of partial fractions decomposition.

Non repeated linear factors.

$$\int \frac{P(x)}{Q(x)}, \text{ degree } P < \text{degree } Q.$$

$Q(x)$ can be factored into a product of non repeated linear factors: $Q(x) = (a_1 x + b_1)(a_2 x + b_2) \cdots (a_n x + b_n)$

Strategy: Find numbers $A_1, A_2, \ldots, A_n$ s.t.

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1 x + b_1} + \frac{A_2}{a_2 x + b_2} + \cdots + \frac{A_n}{a_n x + b_n}.$$

(By equating coefficients or strategic substitution)

Then $$\int \frac{P(x)}{Q(x)}\,dx = \int \frac{A_1}{a_1 x + b_1} + \int \frac{A_2}{a_2 x + b_2} + \cdots + \int \frac{A_n}{a_n x + b_n}$$

Ex: # 13 Find the integral.

$$\int \frac{\sin x}{\cos^2 x + \cos x - 56}\, dx$$

Hint: First, let $u = \cos x$.

Then use partial fractions to find integral after

u-sub.

$u = \cos x$; $du = -\sin x\, dx$

$$\int \frac{-1\, du}{u^2 + u - 56} = \int \frac{-1\, du}{(u-7)(u+8)}$$

$$\frac{-1}{(u-7)(u+8)} = \frac{A}{u-7} + \frac{B}{u+8}$$

$$-1 = A(u+8) + B(u-7)$$

Choose $u = -8$; $-1 = -15B$; $B = 1/15$.

Choose $u = 7$; $-1 = 15A$; $A = -1/15$

$$\int \frac{-1\, du}{(u-7)(u+8)} = \int \frac{-1/15}{u-7} + \int \frac{1/15}{u+8}\, du$$

$$= -\frac{1}{15} \ln|u-7| + \frac{1}{15} \ln|u+8| + C$$

$$= -\frac{1}{15} \ln|\cos x - 7| + \frac{1}{15} \ln|\cos x + 8| + C.$$

Case 2 of Partial Fractions Decomposition

$Q(x)$ has repeated linear factors

E.g. Find $\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx$

deg top > deg bottom → long division first.

$$
\begin{array}{r|l}
 & \boxed{x + 1} \to \text{quotient} \\
x^3 - x^2 - x + 1 & x^4 \qquad - 2x^2 + 4x + 1 \\
 & -(x^4 - x^3 - x^2 + x) \\ \hline
 & \quad x^3 - x^2 + 3x + 1 \\
 & \quad -(x^3 - x^2 - x + 1) \\ \hline
 & \qquad \boxed{4x} \to \text{Remainder}
\end{array}
$$

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx = \int \left(x + 1 + \frac{4x}{x^3 - x^2 - x + 1} \right) dx$$

$$= \boxed{\frac{x^2}{2} + x} + \boxed{\int \frac{4x}{x^3 - x^2 - x + 1}\,dx}.$$

Find $\displaystyle\int \frac{4x}{x^3 - x^2 - x + 1}\,dx$

$$= \int \frac{4x}{x^2(x-1) - (x-1)}\,dx = \int \frac{4x}{(x^2-1)(x-1)}\,dx$$

$$= \int \frac{4x}{(x+1)(x-1)(x-1)}\,dx = \int \frac{4x}{(x+1)(x-1)^2}$$

Repeated linear factor

Partial fractions decomp:

$$\boxed{\frac{4x}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}}$$

(Note: If it were $\frac{4x}{(x+1)(x-1)^3}$, then the decomposition will be:

$$\frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3})$$

$$4x = A(x-1)^2 + B(x-1)(x+1) + C(x+1)$$

Choose $x=1$; $\underbrace{4}_{LHS} = \underbrace{2C}_{RHS} \Rightarrow \boxed{C=2}$

Choose $x=-1$; $\underbrace{-4}_{LHS} = 4A \Rightarrow \boxed{A=-1}$

$$4x = -(x-1)^2 + B(x-1)(x+1) + 2(x+1)$$

Choose $x=2$:

$$8 = -1 + 3B + 6 \Rightarrow \boxed{B=1}$$

$$\Rightarrow \int \frac{4x}{(x+1)(x-1)^2}dx = \int \frac{-1}{x+1} + \int \frac{1}{x-1} + \boxed{\int \frac{2}{(x-1)^2}}$$

$$= \boxed{-\ln|x+1| + \ln|x-1| - \frac{2}{x-1} + C}$$

$$\left(\int \frac{2}{(x-1)^2}dx \; ; \quad u = x-1 \rightarrow \int \frac{2}{u^2}du = 2\int u^{-2}du \right.$$

$$= 2\cdot\frac{u^{-1}}{-1} + C$$

$$= \frac{-2}{u} + C = -\frac{2}{x-1} + C$$