Case 3: $Q(x)$ has irreducible quadratic factors

(quadratic factor that cannot be factored over the real #s)

E.g. (Simple irreducible quadratic factor)

$$\int \frac{dx}{x^2+4x+18} \qquad b^2-4ac = 16 - 4\cdot 18 < 0$$

$$\boxed{\int \frac{du}{u^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C}$$

→ Complete the square method.

$$x^2+4x+18 = \underbrace{x^2+4x+4}_{} - 4 + 18$$
$$= (x+2)^2 + 14$$

$$\int \frac{dx}{(x+2)^2+14} \qquad u = x+2 \longrightarrow \int \frac{du}{u^2+14} = \frac{1}{\sqrt{14}}\cdot \tan^{-1}\left(\frac{x+2}{\sqrt{14}}\right) + C.$$

E.g. $\displaystyle\int \frac{dx}{(x-2)(x^2+2x+4)}$

→ Partial fractions decomposition :

$$\frac{1}{(x-2)(x^2+2x+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4}$$

$$1 = A(x^2+2x+4) + (Bx+C)(x-2)$$

$$1 = Ax^2 + 2Ax + 4A + Bx^2 - 2Bx + Cx - 2C$$

$$1 = (A+B)x^2 + (2A-2B+C)x + 4A-2C$$

$A+B=0$; $2A-2B+C=0$; $4A-2C=1$.

$B=-A$ $\qquad\qquad\qquad\qquad$ $C=\frac{4A-1}{2}$

$$2A+2A+\frac{4A-1}{2}=0$$

$6A-\frac{1}{2}=0$; $\boxed{A=\frac{1}{12}}$; $\boxed{B=-\frac{1}{12}}$; $\boxed{C=-\frac{1}{3}}$

$$\int \frac{1}{(x-2)(x^2+2x+4)}dx = \int \frac{1/12}{x-2}dx + \int \frac{-\frac{1}{12}x - \frac{1}{3}}{x^2+2x+4}dx$$

$$= \frac{1}{12}\ln|x-2| + \boxed{\int \frac{-\frac{1}{12}x - \frac{1}{3}}{x^2+2x+4}dx}$$

$$\int \frac{-\frac{1}{12}x - \frac{1}{3}}{x^2+2x+4}dx \longrightarrow \text{complete the square for denominator.}$$

$$\underbrace{x^2+2x+1}-1+4 = (x+1)^2+3$$

$$\int \frac{-\frac{1}{12}x - \frac{1}{3}}{(x+1)^2+3}dx. \quad \text{let } u = x+1. \quad du = dx$$

$$\Rightarrow x = u-1$$

$$\int \frac{-\frac{1}{12}(u-1) - \frac{1}{3}}{u^2+3}du = \int \frac{-\frac{1}{12}u + \frac{1}{12} - \frac{1}{3}}{u^2+3}du$$

$$= \int \frac{-\frac{1}{12}u - \frac{1}{4}}{u^2+3}du = -\frac{1}{12}\int \frac{u}{u^2+3}du - \frac{1}{4}\int \frac{du}{u^2+3}$$

$w = u^2+3;\ dw = 2u\,du$ $\qquad -\frac{1}{4}\cdot\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)$

$$-\frac{1}{12}\cdot\frac{1}{2}\int \frac{dw}{w}$$

$$-\frac{1}{24}\ln|u^2+3|$$

$$-\frac{1}{24}\ln|(x+1)^2+3| - \frac{1}{4\sqrt{3}}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C$$

Case 4: Repeated irreducible quadratic factors.

E.g. $$\int \frac{2}{(x-4)(x^2+2x+6)^2} = \int \frac{A}{x-4} + \frac{Bx+C}{x^2+2x+6} + \frac{Dx+E}{(x^2+2x+6)^2}$$

E.g. $$\int \frac{1}{(x+1)^3(x-2)(x^2+4)^2(x^2+2x+5)}$$

$\longrightarrow$ Decomposition:

$$\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{D}{x-2} + \frac{Ex+F}{x^2+4} + \frac{Gx+H}{(x^2+4)^2} + \frac{Ix+J}{x^2+2x+5}$$

Ex # 12.

$$\int_0^1 \frac{e^x}{25 - e^{2x}}\, dx \qquad \frac{u = e^x}{du = e^x\, dx}$$

$$\int_1^e \frac{du}{25 - u^2} = \int_1^e \frac{du}{(5-u)(5+u)}$$

$$\frac{1}{(5-u)(5+u)} = \frac{A}{5-u} + \frac{B}{5+u}$$

$$1 = A(5+u) + B(5-u)$$

Choose $u = -5$: $1 = 10B \Rightarrow B = 1/10$

$u = 5$: $1 = 10A \Rightarrow A = 1/10$

$$\int \frac{1/10}{5-u} + \int \frac{1/10}{5+u} = \left(-\frac{1}{10} \ln|5-u| + \frac{1}{10} \ln|5+u|\right)\Bigg|_1^e$$

#11

$$\int \frac{x}{(x-1)(x^2+2x+2)^2}\,dx$$

Partial Fractions Decomposition Form

$$\frac{A}{x-1} + \frac{Bx+C}{x^2+2x+2} + \frac{Dx+E}{(x^2+2x+2)^2}$$

$$\rightarrow \underbrace{\int \frac{1}{25(x-1)}}_{I_1} + \underbrace{\int \frac{-x-3}{25(x^2+2x+2)}}_{I_2} + \underbrace{\int \frac{-x+2}{5(x^2+2x+2)^2}}_{I_3}$$

$$I_1 = \frac{1}{25}\ln|x-1|$$

$$I_2 = \int \frac{-x-3}{25(x^2+2x+2)}\,dx = \frac{-1}{25}\int \frac{x+3}{x^2+2x+2}\,dx$$

$$= -\frac{1}{25}\int \frac{\boxed{x+3}}{(x+1)^2+1}\,dx.$$

let $u = x+1$. $du = dx$.

$x = u-1$

$$= -\frac{1}{25}\int \frac{u-1+3}{u^2+1}\,du = -\frac{1}{25}\int \frac{u+2}{u^2+1}\,du$$

$$= -\frac{1}{25}\left(\int \frac{u}{u^2+1}\,du + 2\cdot\underbrace{\int \frac{du}{u^2+1}}\right)$$

let $w = u^2+1$

$dw = 2u\,du$

$$= -\frac{1}{25}\left(\frac{1}{2}\int \frac{dw}{w} = \frac{1}{2}\ln|w| + \tan^{-1}(u)\right)$$

$$= -\frac{1}{25}\left(\frac{1}{2}\ln|u^2+1| + 2\tan^{-1}(u)\right)$$

$$= -\frac{1}{25}\left(\frac{1}{2}\ln|(x+1)^2+1| + 2\tan^{-1}(x+1)\right)$$

$$I_3 = \int \frac{-x+2}{5(x^2+2x+2)^2}\,dx$$

$$= \frac{1}{5} \cdot \int \frac{-x+2}{(x^2+2x+2)^2}\,dx$$

$$= \frac{1}{5} \cdot \int \frac{-x+2}{\left[(x+1)^2+1\right]^2}\,dx$$

let $u = x+1$; $x = u-1$

$du = dx$

$$= \frac{1}{5} \cdot \int \frac{-(u-1)+2}{(u^2+1)^2}\,du = \frac{1}{5} \cdot \int \frac{-u+3}{(u^2+1)^2}\,du$$

$$= \frac{1}{5} \cdot \int \frac{-u}{(\underbrace{u^2+1}_{w})^2}\,du + \frac{3}{5} \boxed{\int \frac{du}{(u^2+1)^2}}$$

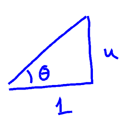
$$\int \frac{du}{(u^2+1)^2} \qquad \boxed{u = \tan\theta} \qquad du = \sec^2\theta\, d\theta$$

$$\int \frac{\sec^2\theta\, d\theta}{(\tan^2\theta+1)^2} = \int \frac{\sec^2\theta}{(\sec^2\theta)^2}\,d\theta$$

$$= \int \frac{1}{\sec^2\theta}\,d\theta = \int \cos^2\theta\, d\theta = \int \frac{1+\cos(2\theta)}{2}\,d\theta$$

$$= \frac{\theta}{2} + \frac{\sin(2\theta)}{4} + C = \frac{\tan^{-1}(u)}{2} + \frac{2\sin\theta\cos\theta}{4}$$

$$= \frac{\tan^{-1}(u)}{2} + \frac{1}{2} \cdot \frac{u}{1+u^2}.$$



$$\cos\theta = \frac{1}{\sqrt{1+u^2}}$$

$$\sin\theta = \frac{u}{\sqrt{1+u^2}}$$