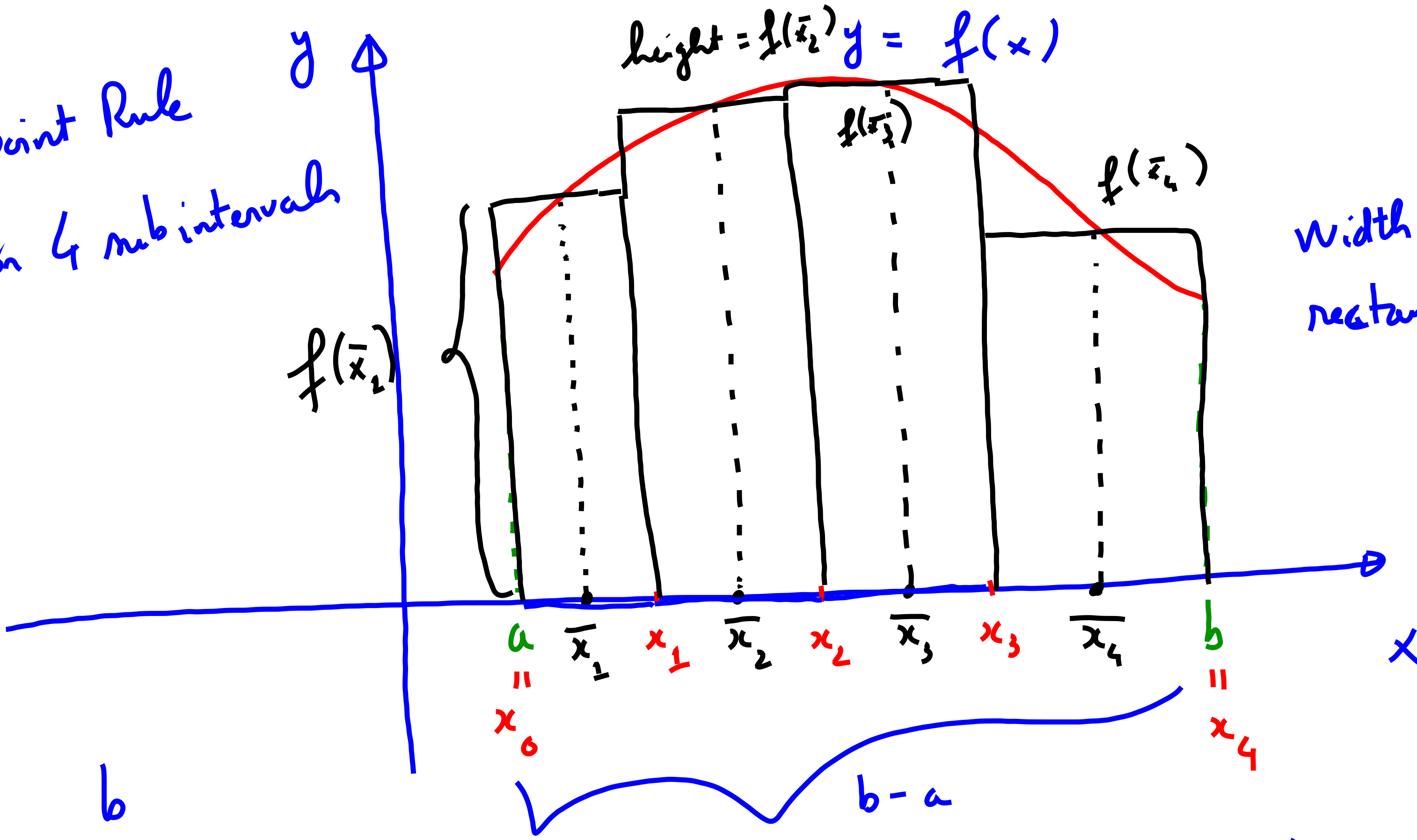
3.6. Numerical Integration.

Goals:

1. Midpoint Rule
2. Trapezoid Rule
3. Simpson's Rule
4. Error estimates for these rules.

Midpoint Rule. $\int_a^b f(x)\,dx \;;\; f(x) \geq 0$

Midpoint Rule for 4 subintervals



width of each rectangle $= \dfrac{b-a}{4} = \Delta x$

$$\int_a^b f(x)\,dx = \text{area under the curve } y = f(x);\ a \le x \le b$$

$$M_4 = \left[f(\bar{x}_1) + f(\bar{x}_2) + f(\bar{x}_3) + f(\bar{x}_4)\right]\Delta x$$

Process and formula for midpoint rule in general.

(1) Subdivide $[a,b]$ into n subintervals of equal length.

length of each subinterval:

$$\Delta x = \frac{b-a}{n}.$$

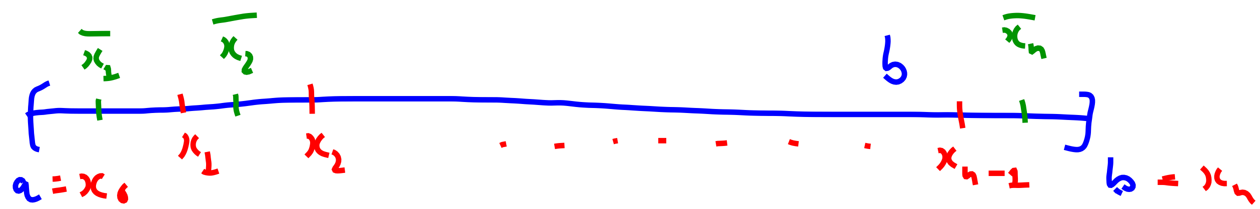
(2) Find the sum:

$$M_n = \left[f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_n) \right] \cdot \Delta x.$$

M_n is the midpoint approximation using n subintervals.

Here $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n$ are the midpoints of the interval

$[\underbrace{x_0}_{a}, x_1]; [x_1, x_2], \ldots, [x_{n-1}, \underbrace{x_n}_{b}]$.

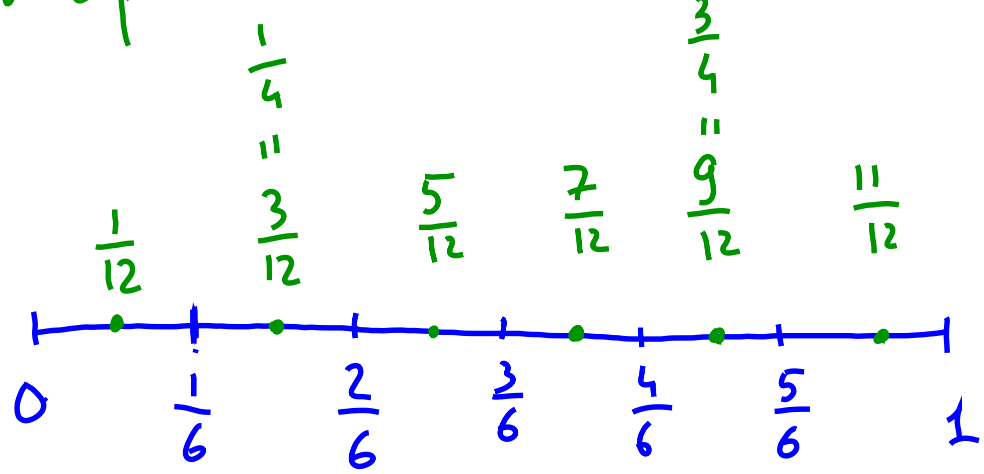


E.g. $\int_0^1 \ln(x+5)\,dx$

Use midpoint rule $n = 6$ to estimate this integral.

Length of each subinterval:

$$\frac{1-0}{6} = \frac{1}{6}.$$

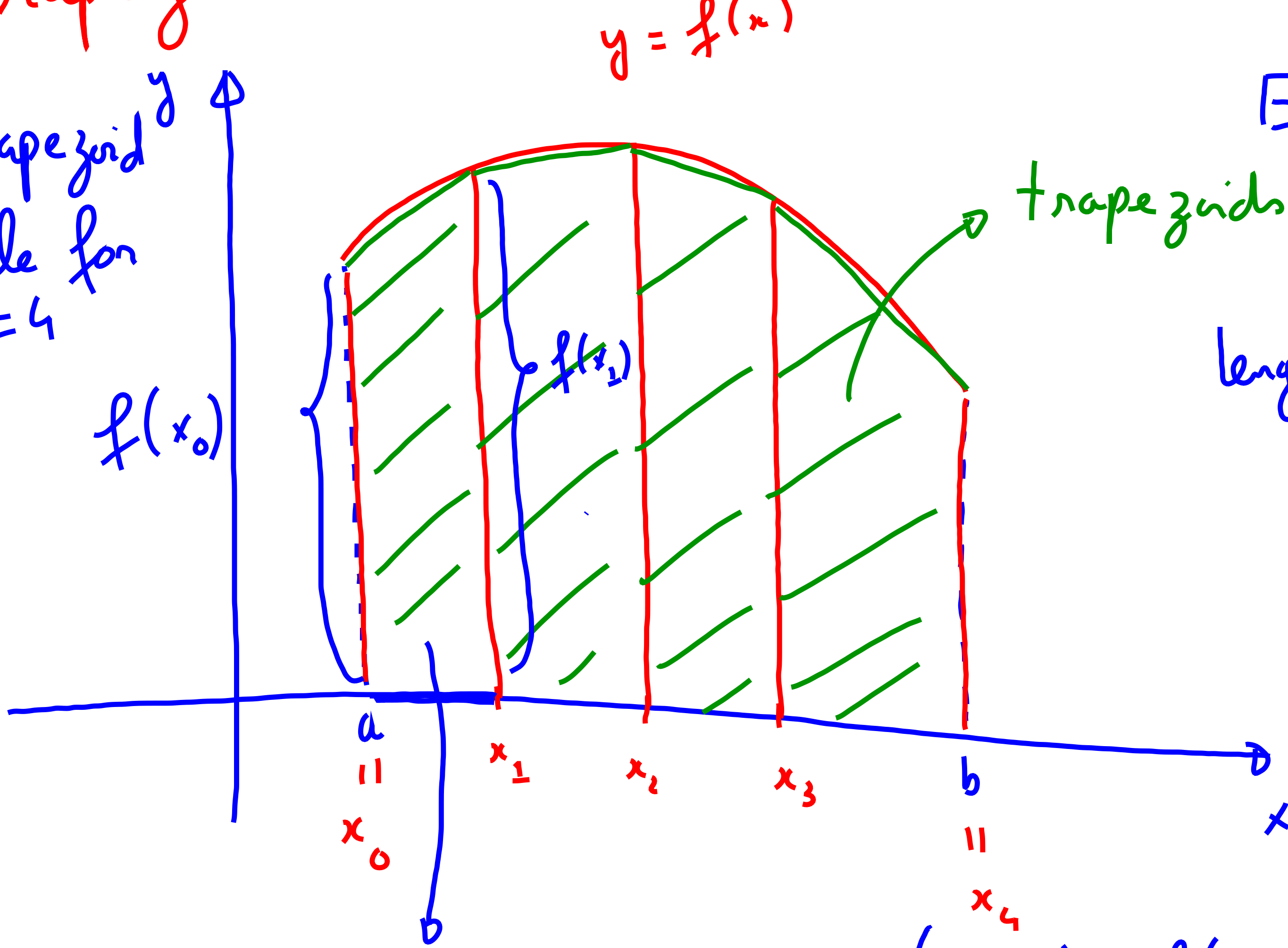


$$\frac{\frac{1}{6}+\frac{2}{6}}{2} \qquad \frac{\frac{2}{6}+\frac{3}{6}}{2}$$

$$M_6 = \left[f\left(\frac{1}{12}\right) + f\left(\frac{1}{4}\right) + f\left(\frac{5}{12}\right) + f\left(\frac{7}{12}\right) + f\left(\frac{3}{4}\right) + f\left(\frac{11}{12}\right) \right] \cdot \frac{1}{6} \approx 1.7034$$

Trapezoid Rule

Trapezoid Rule for $n=4$



Estimate $\int_a^b f(x)\,dx$

length of a sub interval:

$$\frac{b-a}{4} = \Delta x$$

Area of this trapezoid $= \dfrac{(f(x_0) + f(x_1)) \cdot \Delta x}{2}$

Sum of the area of these 4 trapezoid:

$$\frac{\Delta x}{2}\Big[f(x_0) + f(x_1) + f(x_1) + f(x_2) + f(x_2) + f(x_3) + f(x_3) + f(x_4)\Big]$$

$$\frac{\Delta x}{2}\Big[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\Big] = T_4$$

In general, the formula for Trapezoid Rule is:

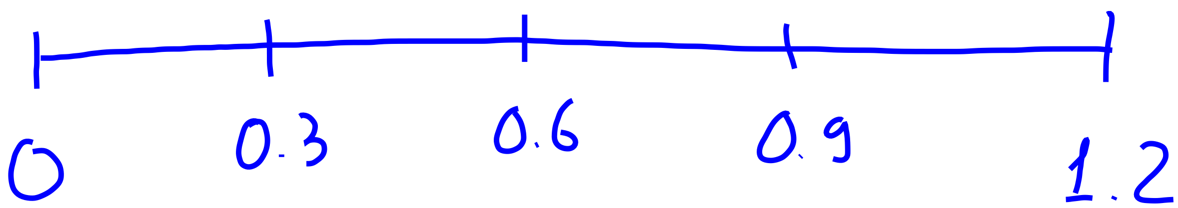
$$\boxed{T_n = \frac{\Delta x}{2}\Big[f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n) \Big]}$$

where $\Delta x = \dfrac{b-a}{n}$ = length of each subinterval.

E.g. $\displaystyle\int_0^{1.2} \sin(x^2)\,dx$. Estimate this integral with trapezoid rule for $n = 4$

length of each subinterval

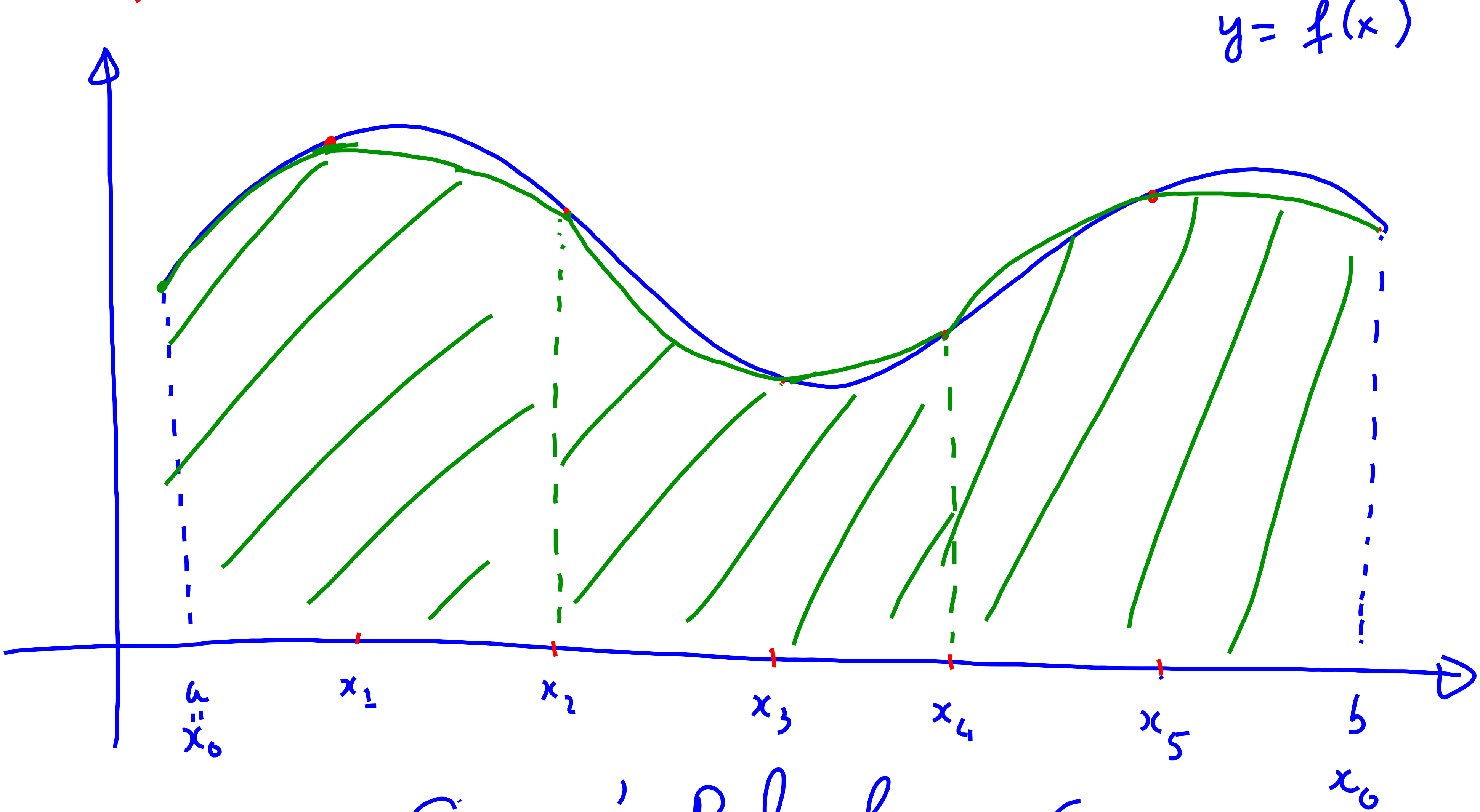
$$\Delta x = \frac{1.2 - 0}{4} = 0.3$$



$$T_4 = \frac{0.3}{2}\Big(f(0) + 2f(0.3) + 2f(0.6) + 2f(0.9) + f(1.2) \Big) \approx 0.498651$$

Here: $f(x) = \sin(x^2)$

Simpson's Rule



Simpson's Rule for $n = 6$

$$\Delta x = \frac{b-a}{6}$$

$$S_6 = \frac{\Delta x}{3}\Big(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6)\Big)$$

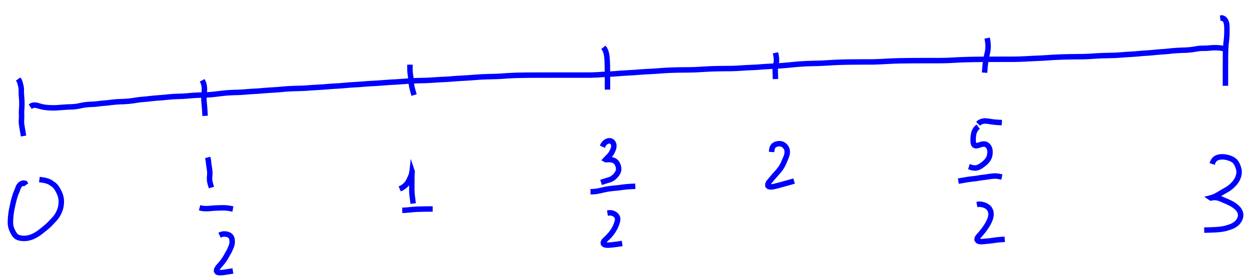
In general, the formula for Simpson's Rule is:

$$S_n = \frac{\Delta x}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \Big].$$

$\Delta x = \dfrac{b-a}{n}$. ($n$ has to be an even number)

<u>E.g.</u> $\displaystyle\int_0^3 \frac{1}{1+x^3}\,dx$. Use Simpson's Rule, to estimate with $n=6$ this integral.

$$\Delta x = \frac{3-0}{6} = \frac{1}{2}$$



$$S_6 = \frac{\frac{1}{2}}{3}\Big[f(0) + 4f(\tfrac{1}{2}) + 2f(1) + 4f(\tfrac{3}{2}) + 2f(2) + 4f(\tfrac{5}{2}) + f(3) \Big] \approx 1.1614$$

Error estimates.

Estimate $\displaystyle\int_a^b f(x)\,dx$

using Midpoint Rule with n subintervals : M_n

—— Trapezoid Rule ———— : T_n

—— Simpson's Rule ———— : S_n

Exact errors for each of these methods:

Error for Midpoint : $E_M = \displaystyle\int_a^b f(x)\,dx - M_n$

———— Trapezoid : $E_T = \displaystyle\int_a^b f(x)\,dx - T_n$

———— Simpson : $E_S = \displaystyle\int_a^b f(x)\,dx - S_n$

Error bounds.

We have: $|E_M| \leq \boxed{\dfrac{K(b-a)^3}{24n^2}}$ → upper bound for the error in Midpoint

$$|E_T| \leq \boxed{\frac{K(b-a)^3}{12n^2}}$$ → upper bound for the error in Trapezoid.

where K is a number such that $|f''(x)| \leq K$ for all x in $[a,b]$.

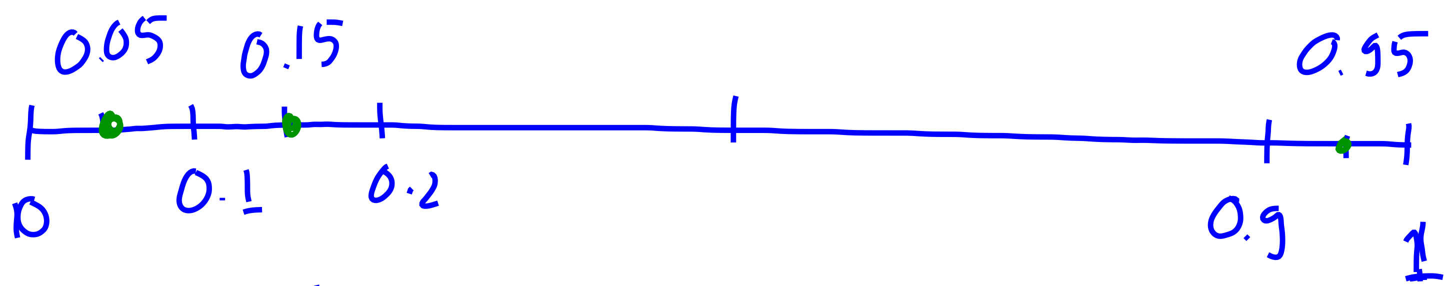
For Simpson: $|E_S| \leq \boxed{\dfrac{K(b-a)^5}{180n^4}}$

where K is a number such that $|f^{(4)}(x)| \leq K$ for all x in $[a,b]$.

E.g. $\displaystyle\int_0^1 e^{x^2}\,dx$

1. Use the midpoint rule for $n = 10$ to estimate this integral
2. Find an upper bound for the error involved in this approximation.

1.



$$\Delta x = \frac{1-0}{10} = 0.1$$

$$M_{10} = \Delta x\left[f(0.05) + f(0.15) + \dots + f(0.95)\right]$$
$$\approx 1.460393$$

2. Find an upper bound for E_M.

$$|E_M| \le \frac{K(b-a)^3}{24n^2}$$

$n = 10$; $b = 1$; $a = 0$

K is an upper bound for $|f''(x)|$

$f(x) = e^{x^2}$; $f'(x) = 2xe^{x^2}$

$$f''(x) = 2\cdot\left[e^{x^2} + x\cdot 2xe^{x^2}\right]$$

$$= \boxed{2\cdot \boxed{e^{x^2}}\left[1+2x^2\right]}$$

What is an upper bound for $|f''(x)|$ on $[0,1]$?

Since f'' is an increasing function on $[0,1]$, the largest it could be is the value at 1

$f''(1) = 2\cdot e\cdot(1+2) = 6e \rightarrow K$

$$|E_M| \le \frac{6e \cdot (1-0)}{24 \cdot 10^2} \approx \boxed{0.0067}$$

the error cannot be more than this

E.g. Using error bound formula for Simpson's Rule.

How large should we take n in order to guarantee that the Simpson approximation for $\int_1^2 \frac{1}{x}\,dx$ is accurate to within 0.0001?

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}$$

where K is an upper bound for $|f^{(4)}(x)|$

$b=2$; $a=1$; $n \longrightarrow$ don't know.

$f(x)=\frac{1}{x}$; $f'(x) = -\frac{1}{x^2}$; $f''(x) = \frac{2}{x^3}$

$f^{(3)}(x) = \frac{-6}{x^4}$; $|f^{(4)}(x)| = \left|\frac{24}{x^5}\right| \longrightarrow$ decreasing function.

On $[1,2]$, $f^{(4)}(x)$ will be largest at $x=1$; $f^{(4)}(1) = \boxed{24}$

Upper bound for the error

$$\frac{24 \cdot (2-1)^5}{180 n^4} \leq 0.0001$$

$$\frac{24}{180 n^4} \leq \boxed{0.0001}$$

$$24 \leq \boxed{(0.0001) \cdot 180} n^4$$

$$1333.33 = \frac{24}{(0.0001)(180)} \leq n^4$$

$$n \geq 6.0427$$

$\boxed{n \text{ is } 7}$ or more