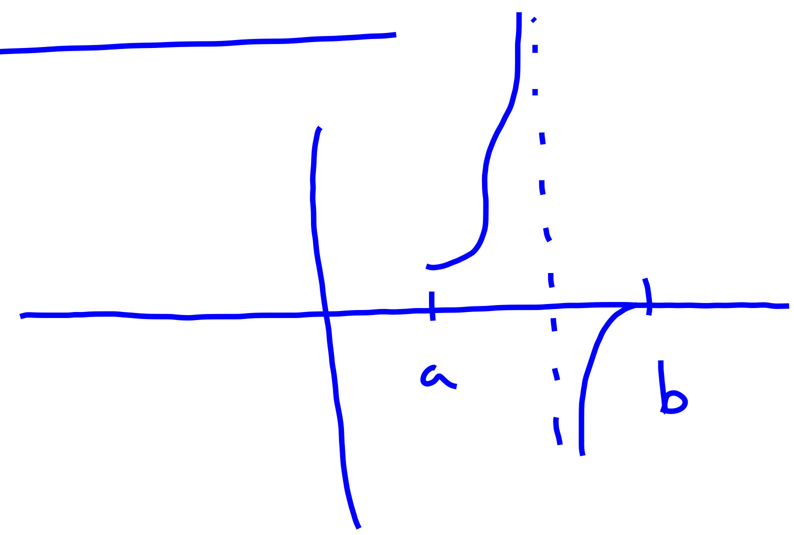
3.7. Improper Integrals

$\int_a^b f(x)\,dx$: a, b: finite #'s.

f is continuous over $[a,b]$

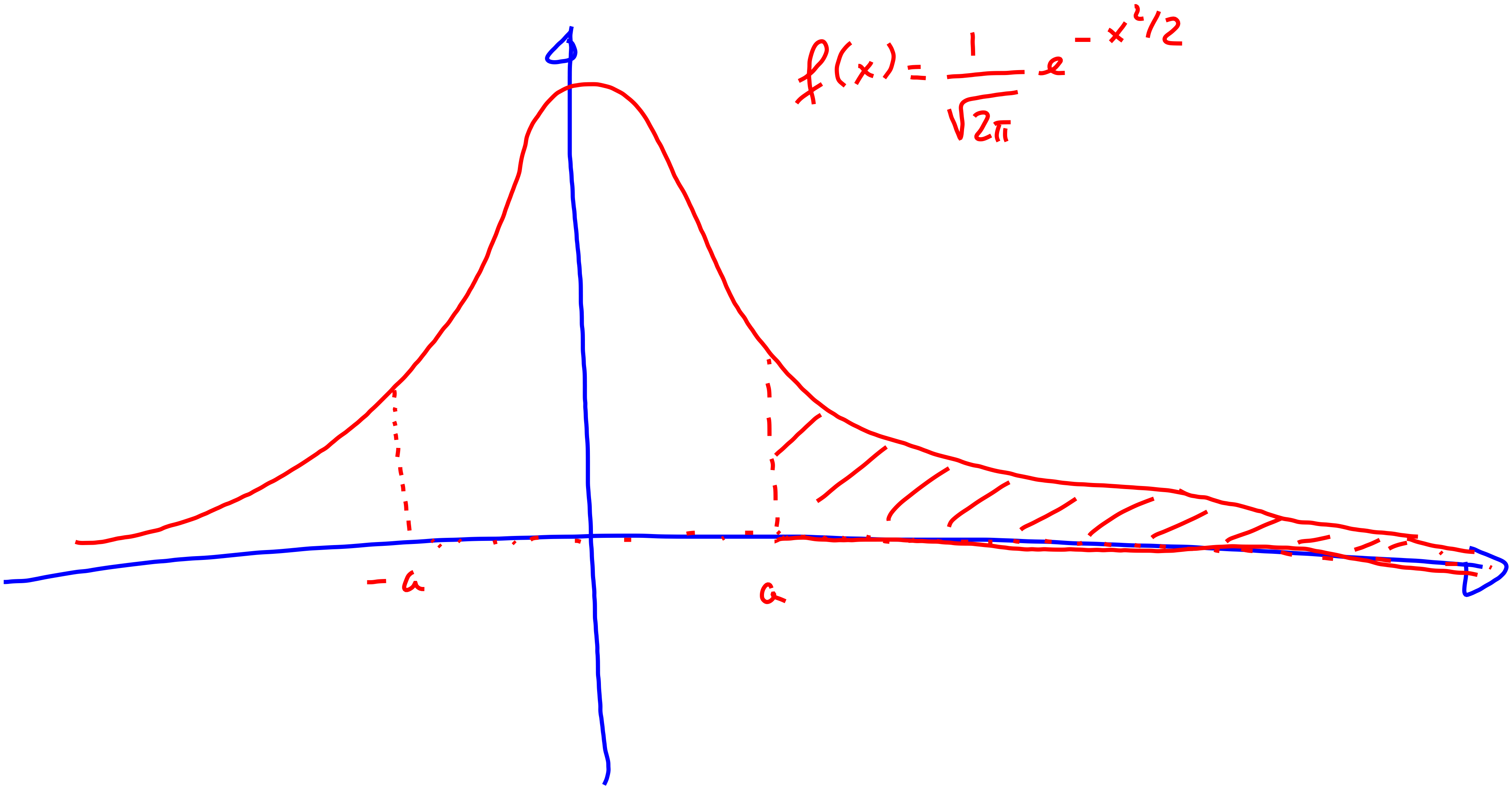
} Regular (Proper Integrals



What if $a = \infty$ or $-\infty$

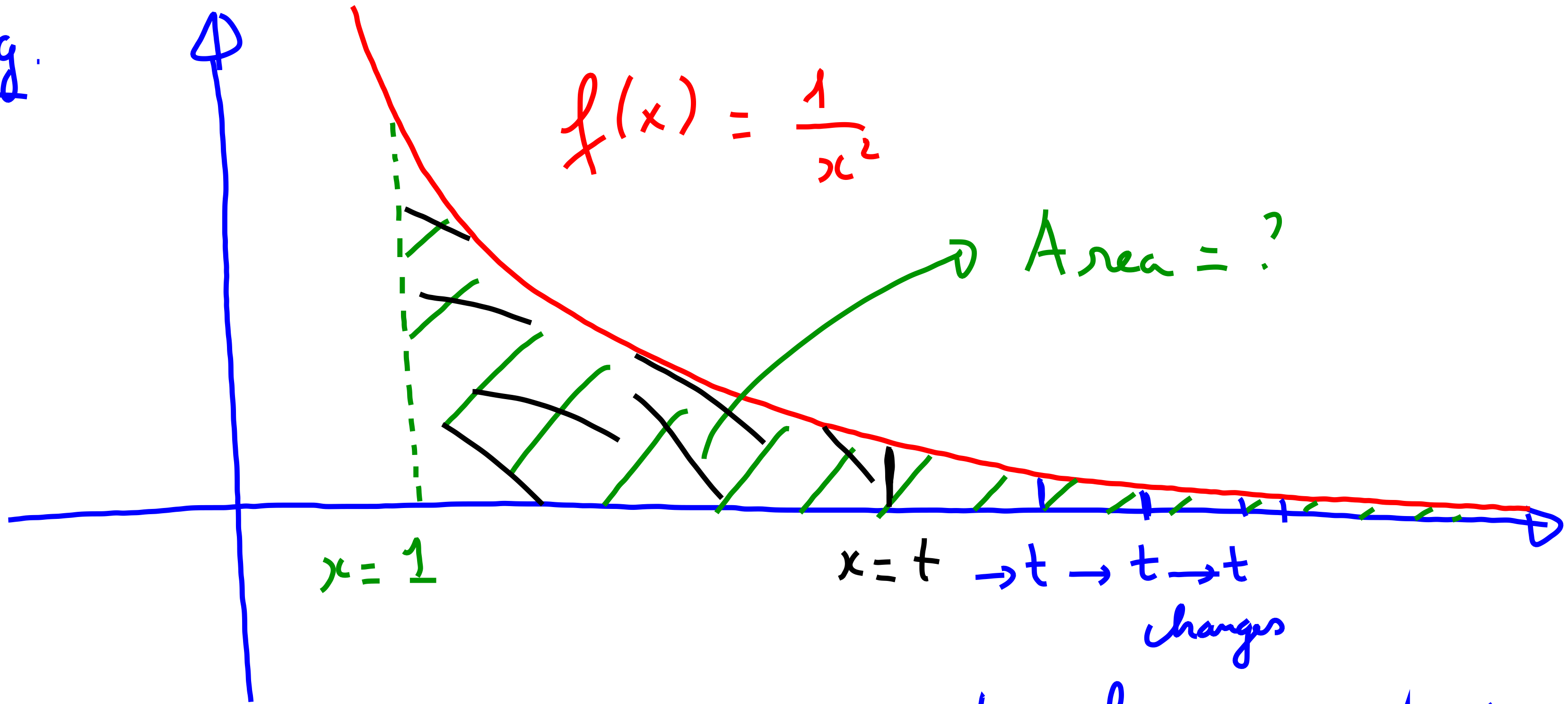
or $b = \infty$ or $-\infty$.

or f is not cont. on $[a,b]$.



$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$
$-a$
a

E.g.



Area under graph of $y = \frac{1}{x^2}$ from $x = 1$ to $x = t$

$$\int_1^t \frac{1}{x^2}\,dx = \int_1^t x^{-2}\,dx = -\frac{1}{x}\Big|_1^t = -\frac{1}{t} + 1$$

$$A(t) = 1 - \frac{1}{t}.$$

Area under $y = \frac{1}{x^2}$; $x = 1$ to ∞

$$= \lim_{t \to \infty} \left(1 - \frac{1}{t} \right) = 1$$

Type 1 Improper integral : Infinite integrals

$$\int_a^\infty f(x)\,dx := \lim_{t \to \infty} \int_a^t f(x)\,dx$$

provided that the limit exists (as a finite number)

If it does exist, we say that $\int_a^\infty f(x)\,dx$ converges

$$* \quad \int_{-\infty}^{b} f(x)\,dx := \lim_{t \to -\infty} \int_{t}^{b} f(x)\,dx.$$

provided that the limit exists as a finite #.

$$* \quad \int_{-\infty}^{\infty} f(x)\,dx := \underbrace{\int_{-\infty}^{a} f(x)\,dx}_{\lim_{t \to -\infty} \int_{t}^{a} f(x)\,dx} + \underbrace{\int_{a}^{\infty} f(x)\,dx}_{\lim_{t \to \infty} \int_{a}^{t} f(x)\,dx}.$$

a: any real #

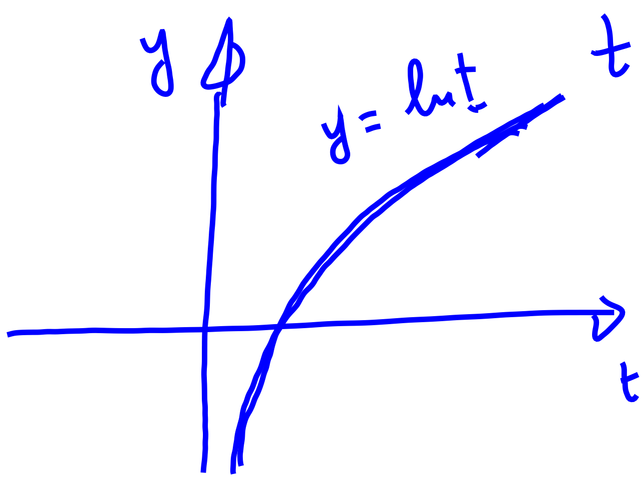
$$\lim_{t \to -\infty} \int_{t}^{a} f(x)\,dx + \lim_{t \to \infty} \int_{a}^{t} f(x)\,dx$$

E.g. $\displaystyle\int_1^\infty \frac{1}{x}\,dx$.

Does this integral converge?

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty}\left(\ln x \Big|_1^t\right)$$

$$= \lim_{t\to\infty}\left(\ln t - \ln 1\right) = \lim_{t\to\infty}(\ln t)$$



$$= \infty$$

it does not converge.

Ex. Does $\int_{-\infty}^{0} x e^{x}\,dx$ converge?

If it does, find the # that it converges to.

(Hint: L'Hopital Rule)

$\frac{0}{0}$; $\frac{\infty}{\infty}$

p- integral :

$$\int_1^{\infty} \frac{1}{x^p}\,dx$$

if $p > 1$, this integral converges

if $p \leq 1$, this integral diverges.

We've seen that $p = 1$: $\int_1^{\infty} \frac{1}{x}\,dx$ diverges.

Consider : $p \neq 1$.

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \int_1^{\infty} x^{-p}\,dx = \lim_{t \to \infty} \int_1^{t} x^{-p}\,dx$$

$$= \lim_{t \to \infty} \left. \frac{x^{-p+1}}{-p+1} \right|_1^t = \lim_{t \to \infty} \frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1}.$$

$$= \frac{1}{1-p} \left[\lim_{t \to \infty} \left(t^{1-p} - 1 \right) \right]$$

$$= \frac{1}{1-p} \left[\lim_{t \to \infty} \left(\frac{1}{t^{p-1}} - 1 \right) \right.$$

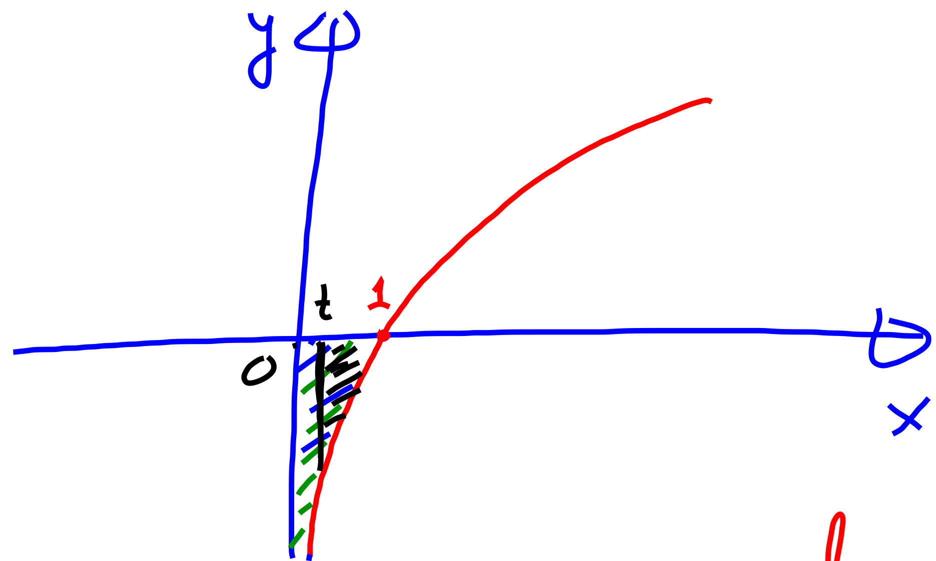
$p > 1$: converges

$p < 1$: diverges.

Type 2 Improper Integral:

Integral of discontinuous function

E.g. $\displaystyle\int_0^1 \ln x \, dx$



$$= \lim_{t \to 0^+} \boxed{\int_t^1 \ln x \, dx}$$

t approaches 0 from the right

$$\begin{cases} u = \ln x \\ dv = dx \end{cases} \Rightarrow \begin{cases} du = \frac{1}{x}\,dx \\ v = x \end{cases}$$

$$= \lim_{t \to 0^+} \left(x \ln x \Big|_t^1 - \int_t^1 dx \right)$$

$$= \lim_{t \to 0^+} \left(-t \ln t - x \Big|_t^1 \right)$$

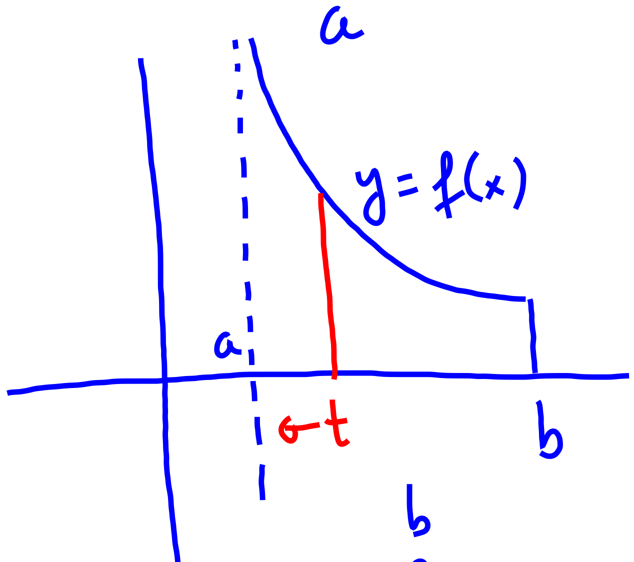
$$= \lim_{t \to 0^+} \Big(\underbrace{-\overset{\to 0}{t}\,\overset{\to -\infty}{\ln t}}_{\to 0} - 1 + \overset{\to 0}{t} \Big) = -1$$

$$-\lim_{t \to 0^+} t \ln t = -\lim_{t \to 0^+} \frac{\ln t}{\frac{1}{t}} = -\lim_{t \to 0^+} \frac{\frac{1}{t}}{-\frac{1}{t^2}}$$

$$= -\lim_{t \to 0^+} \frac{1}{t} \cdot \left(-t^2\right) = \lim_{t \to 0^+} t = 0$$

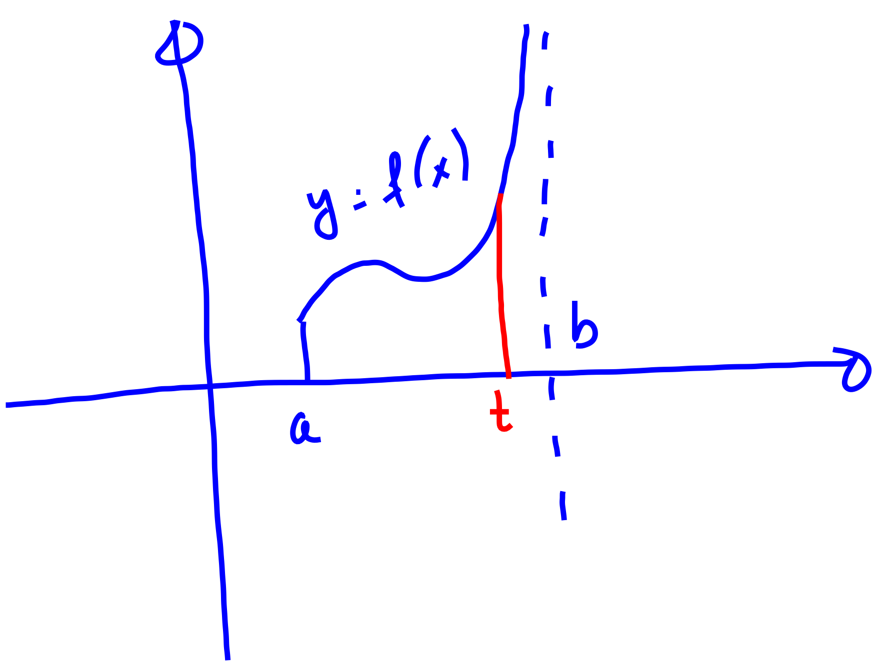
Type 2 improper integral.

* $\int_a^b f(x)\,dx$; $f(x)$ is discontinuous at a.



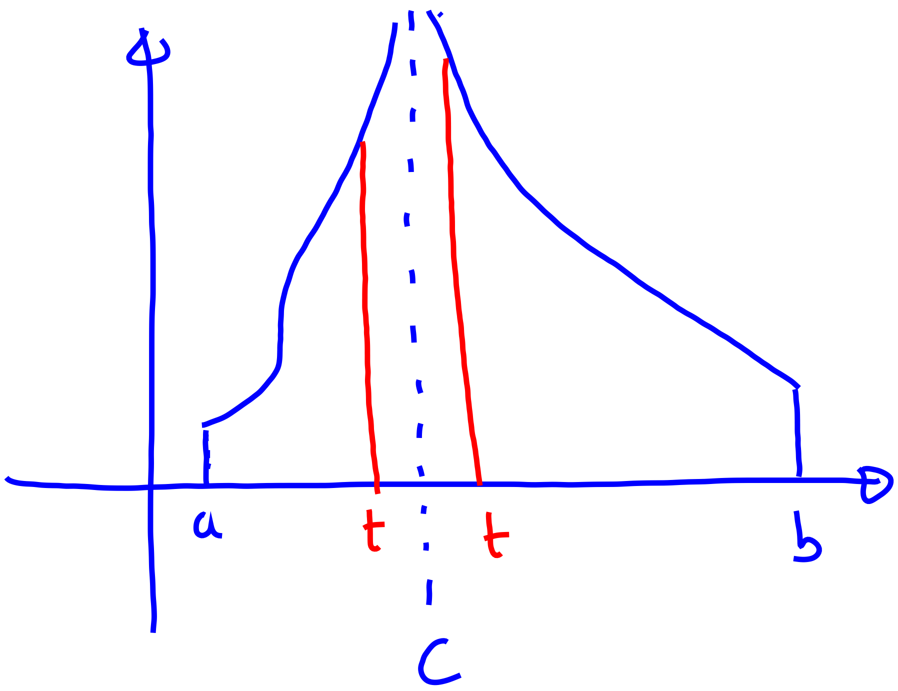
$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

* $\int_a^b f(x)\,dx$; $f(x)$ is discontinuous at b



$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

* $\int_a^b f(x)\,dx$; $f(x)$ is discontinuous at a point c within (a,b)



$$\int_a^b f(x) = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$\lim_{t\to c^-}\int_a^t f(x)\,dx + \lim_{t\to c^+}\int_t^b f(x)\,dx$$