5.1. Sequences

A sequence is a list of numbers with a particular order.

E.g. $2, 4, 8, 16, 32, \ldots$

In general, the numbers in a sequence are denoted by

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

a_1 : first term, a_2 : second term, …, a_n : n^{th} term or general term.

the term immediately after a_n is denoted by a_{n+1}

____________ before a_n is denoted by a_{n-1}

Notation for the sequence $\{a_1, a_2, \ldots\}$; $\{a_n\}$: $\{a_n\}_{n=1}^{\infty}$

E.g. A sequence given by a formula:

$$a_n = \boxed{(-1)^{n-1}} \cdot \frac{n+2}{5^n}$$

$$a_1 = (-1)^0 \cdot \frac{3}{5} = \frac{3}{5} \quad ; \quad a_2 = (-1)^1 \cdot \frac{4}{25} = -\frac{4}{25} \quad ; \quad a_3 = \frac{5}{125}$$

E.g. $\left\{ -\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \frac{5}{81}, \dots \right\}$

Find a formula for the general term a_n of this sequence.

$$a_n = ? \qquad \frac{n+1}{(-3)^n} \quad ; \quad a_n = (-1)^n \cdot \frac{n+1}{3^n}$$

$$a_1 = \boxed{-\frac{2}{3}} \quad ; \quad a_2 = \boxed{\frac{3}{9}} \quad ; \quad a_3 = \boxed{-\frac{4}{27}}$$

E.g. Sequence is given recursively.

Fibonacci sequence $\quad a_1 = 1 \; ; \; a_2 = 1 \; ; \; a_3 = 2 \; ; \; a_4 = 3 \; ;$

$a_5 = 5 \; ; \; a_6 = 8 \; ; \; \ldots$

Recursive formula: $\quad a_n = a_{n-1} + a_{n-2} \; ; \; n \geq 3$

$a_1 = a_2 = 1.$

Limit of a sequence.

$$a_n = f(n)$$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} f(n)$$

E.g. $a_n = \dfrac{3n^2+n+4}{4n^2-3}$

Find $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{3n^2+n+4}{4n^2-3}$.

1st way: $\lim_{n\to\infty} \dfrac{\frac{3n^2+n+4}{n^2}}{\frac{4n^2-3}{n^2}} = \lim_{n\to\infty} \dfrac{3+\frac{1}{n}+\frac{4}{n^2}}{4-\frac{3}{n^2}} = \boxed{\dfrac{3}{4}}$

$\frac{1}{n} \to 0$, $\frac{4}{n^2} \to 0$, $\frac{3}{n^2} \to 0$

2nd way: $\lim_{n\to\infty} \dfrac{\boxed{3n^2}+n+4}{4n^2-3} \approx \dfrac{3\cancel{n^2}}{4\cancel{n^2}} = \boxed{\dfrac{3}{4}}$

$\approx$ behaves like (when n is large)

$\underline{3^{rd} \text{ way}}$: L'Hopital Rule . $\frac{\infty}{\infty}$ → L'Hopital Rule applies

$$\lim_{n\to\infty} \frac{3n^2+n+4}{4n^2-3} = \lim_{n\to\infty} \frac{6n+1}{8n} \overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty} \frac{6}{8} = \boxed{\frac{3}{4}}$$

$\underline{\text{E.g.}}$ $\quad b_n = 4\sqrt{n}\,\ln\left(1+\frac{1}{n}\right)$

Find $\lim_{n\to\infty} b_n$?

$$\lim_{n\to\infty} 4\sqrt{n}\,\ln\left(1+\frac{1}{n}\right) = 4\cdot \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{\sqrt{n}}} \overset{\frac{0}{0}}{=}$$

$\underbrace{\infty \cdot 0}_{\text{indeterminate form}}$

$\ln\left(1+\frac{1}{n}\right) \to 0$

$\frac{1}{\sqrt{n}} \to 0$

L'Hopital will apply.

$$= 4 \cdot \lim_{n\to\infty} \frac{\boxed{\dfrac{-\frac{1}{n^2}}{1+\frac{1}{n}}}}{\boxed{-\frac{1}{2\sqrt{n}\cdot n}}} = 4 \cdot \lim_{n\to\infty} \frac{\frac{1}{n^2}\cdot 2n\sqrt{n}}{1+\frac{1}{n}}$$

$$= 4 \cdot \lim_{n\to\infty} \frac{\frac{2\sqrt{n}}{n}}{1+\frac{1}{n}} = 4 \cdot \lim_{n\to\infty} \frac{\frac{2}{\sqrt{n}}}{1+\frac{1}{n}} = 4 \cdot \frac{0}{1+0} = 0$$

$\frac{2}{\sqrt{n}} \to 0$, $\frac{1}{n} \to 0$

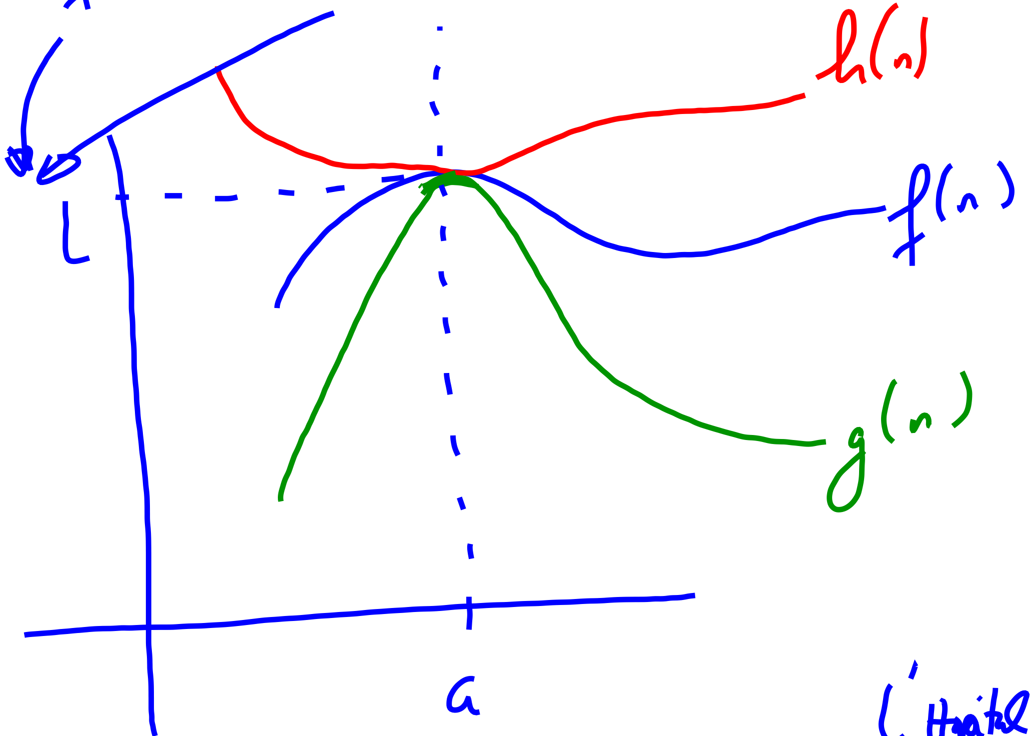
E.g. $a_n = n \cdot \sin\left(\frac{1}{n}\right)$

$\lim_{n\to\infty} a_n = ?$ $\qquad \lim_{n\to\infty} n \cdot \sin\left(\frac{1}{n}\right)$

$$\lim_{n\to\infty} n \cdot \sin\left(\frac{1}{n}\right)$$

Squeeze Theorem ?

$$g(n) \le f(n) \le h(n)$$



$$-1 \le \sin\left(\frac{1}{n}\right) \le 1$$

$$-n \le n \sin\left(\frac{1}{n}\right) \le n$$

$-n \to -\infty$ and $n \to \infty$

Doesn't work.

$\longrightarrow$ L'Hopital.

$$\lim_{n\to\infty} n \cdot \sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \frac{0}{0} \to \text{L'Hopital applies}$$

$$\infty \cdot 0$$

$$= \lim_{n\to\infty} \frac{-\cancel{\frac{1}{n^2}} \cdot \cos\left(\frac{1}{n}\right)}{-\cancel{\frac{1}{n^2}}} = \lim_{n\to\infty} \cos\left(\boxed{\frac{1}{n}}\right) = \cos(0) = \boxed{1}$$

$$\frac{1}{n} \to 0$$

<u>E.g.</u> where the Squeeze Theorem will work

$$a_n = 2^{-n}\cos(n\pi) = \frac{\cos(n\pi)}{2^n}$$

$$\left\{ -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \dots \right\}$$

$$\lim_{n\to\infty} a_n = 0.$$

<u>Using Squeeze Theorem:</u> $-1 \le \cos(n\pi) \le 1$

$$-\frac{1}{2^n} \le \frac{\cos(n\pi)}{2^n} \le \frac{1}{2^n}$$

$$0$$

Definition:

A sequence $\{a_n\}_{n=1}^{\infty}$ is called monotonic if it is either an increasing sequence or a decreasing sequence.

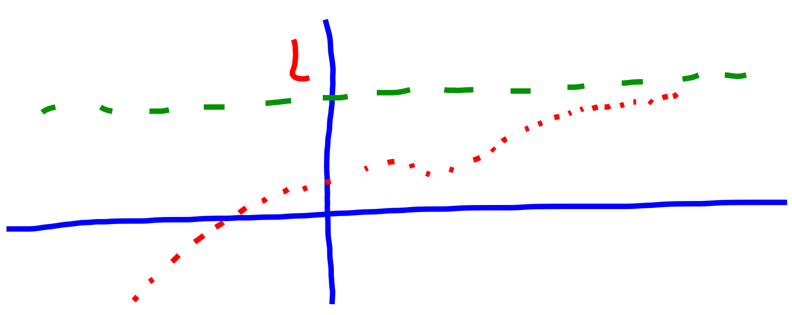
E.g.

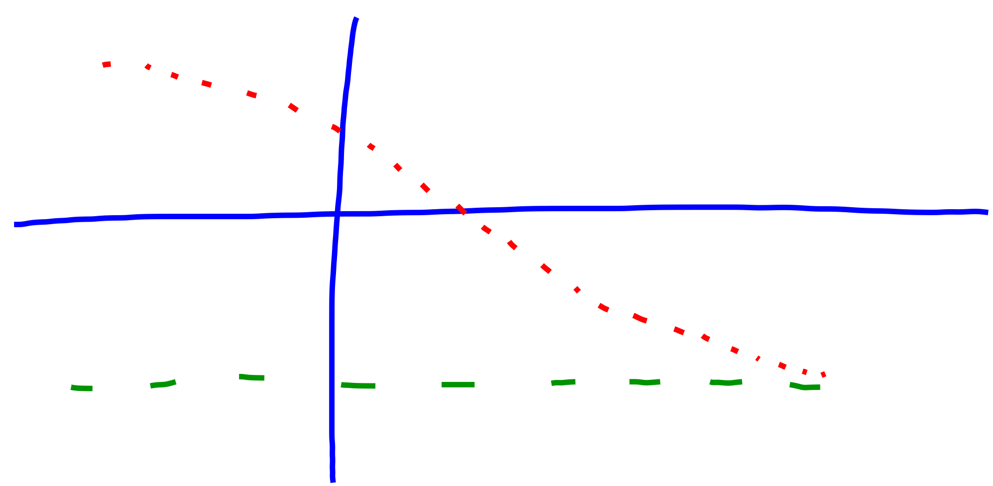
$a_n = n$: an increasing sequence

$a_n = \frac{1}{n}$: a decreasing sequence

$\Big\}$ there are monotonic sequences.

Fact: Every bounded, monotonic sequence will have a limit.





E.g. $a_n = \dfrac{n}{n^2+1}$.

Is $\{a_n\}$ monotonic? Yes, it is a decreasing sequence.
(b/c deg bottom > deg top)

Another, more general way to look at this is to find $f'(n)$

$$f(n) = \frac{n}{n^2+1} \; ; \; f'(n) = \frac{1-n^2}{(n^2+1)^2} < 0$$

when $n > 1$

$f'(n) < 0 \implies f(n)$ is decreasing.

a_n is bounded below by zero $\implies$ it has a limit.