Divergence Test and The Integral Test

Divergence Test.

Suppose we have a series $\sum_{n=1}^{\infty} a_n$.

If $\lim_{n\to\infty} a_n$ does not exist or $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

E.g. $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ $\quad a_n$

$a_n = \frac{n^2}{5n^2+4}$. $\quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{5n^2+4}$

$= \frac{1}{5} \neq 0$

So, the series diverges by the Divergence Test.

E.g. $\sum_{n=1}^{\infty} e^{1/n^2}$

$\boxed{a_n = e^{1/n^2}}$ $\quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{1/n^2} = e^{0} = 1 \neq 0$

$1/n^2 \to 0$

So, the series is divergent.

E.g. $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n^2}\right)$

$a_n = \cos\left(\frac{1}{n^2}\right)$. $\quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos\left(\frac{1}{n^2}\right) = \cos(0) = 1 \neq 0$

So, the series is divergent.

Warning: If $\lim_{n\to\infty} a_n = 0$, the test fails, that is, it does not give us any information.

E.g. $\sum_{n=1}^{\infty} \frac{1}{n}$: harmonic series. It is known to be divergent.

$\lim_{n\to\infty} \frac{1}{n} = 0$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$. This series is known to be convergent

$$\lim_{n \to \infty} \frac{1}{n^2} = 0$$

The integral test

Suppose $\sum_{n=1}^{\infty} a_n$ is a series. Suppose $a_n = f(n)$.

If all the following conditions are satisfied

1. f is positive on $[N, \infty)$ for some $N \geq 1$
2. f is continuous on $[N, \infty)$
3. f is decreasing on $[N, \infty)$ (i.e., $f' < 0$ on $[N, \infty)$)

Then :

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_N^{\infty} f(x)\,dx$$

both converge or both diverge.

In other words, if $\int_N^{\infty} f(x)\,dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

if $\int_N^{\infty} f(x)\,dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

E.g.

(1) $\sum_{n=1}^{\infty} \frac{1}{n}$. $\quad a_n = \frac{1}{n} = f(n)$. $\quad f(x) = \frac{1}{x}$.

* Is f positive on $[1,\infty)$? Yes.
* Is f continuous on $[1,\infty)$? Yes

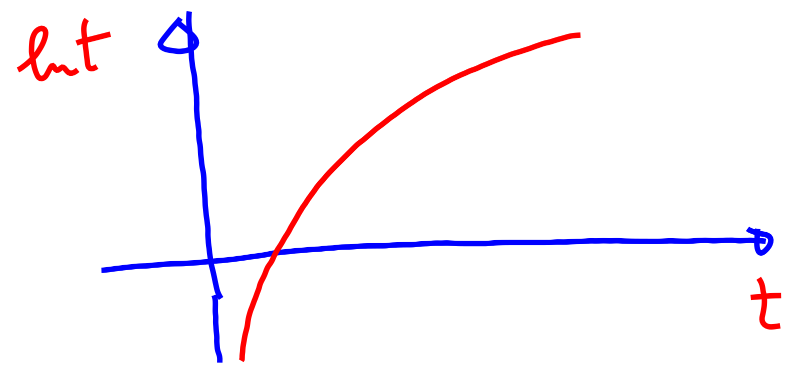
* Is f decreasing on $[1, \infty)$?

$$f'(x) = -\frac{1}{x^2} < 0 \quad \text{on } [1, \infty)$$

Hence, f is decreasing on $[1, \infty)$

Consider $$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{t \to \infty} \int_1^t \frac{1}{x}\,dx = \lim_{t \to \infty} \left(\ln x\right)\Big|_1^t$$

$$= \lim_{t \to \infty} \left(\ln t - \cancelto{0}{\ln 1}\right) = \lim_{t \to \infty} \left(\ln t\right) = \infty$$



So, $\int_1^{\infty} \frac{1}{x}\,dx$ diverges. Hence, the integral test says that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

E.g. Consider $\sum_{n=1}^{\infty} \frac{1}{n^2}$. $a_n = \frac{1}{n^2} = f(n)$ where $f(x) = \frac{1}{x^2}$.

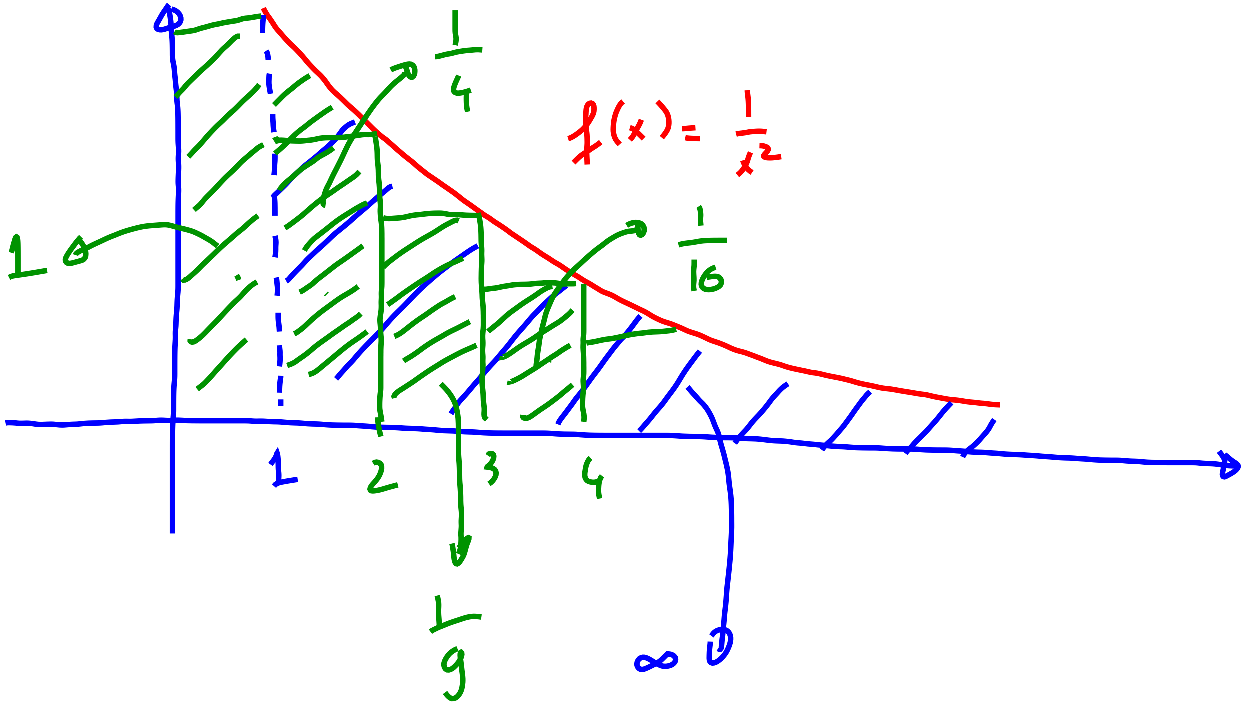
* Is f positive on $[1,\infty)$? Yes.
* Is f continuous on $[1,\infty)$? Yes.
* Is f decreasing on $[1,\infty)$? Yes.

$$f'(x) = \frac{-2}{x^3} < 0 \text{ on } [1,\infty).$$

Consider $\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \left(-\frac{1}{x}\right)\Big|_1^t$

$$= \lim_{t\to\infty} \left(-\frac{1}{t} + 1\right) = 1.$$

So, $\int_1^{\infty} \frac{1}{x^2}\,dx$ converges. Hence, by the Integral Test, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.



$$\int_1^\infty \frac{1}{x^2}\,dx = \text{Area under curve from 1 to } \infty$$

<u>Note:</u> Even though the integral tells us whether the series converges or diverges, the value of the integral and the sum of the series are two very different things.

In general,

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

The Integral Test tells us that this series converges if $p > 1$ and diverges if $p \leq 1$.

E.g. $$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}. \quad p = \frac{1}{3} \longrightarrow \text{series diverges}$$

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}. \quad p = \frac{3}{2} \longrightarrow \text{series converges.}$$

These series are called p-series.

HW # 4

Is $\sum_{n=2}^{\infty} \frac{7}{n\sqrt{\ln n}}$ diverges or converges ?

The function which gives the terms of the series is

$$f(x) = \frac{7}{x\sqrt{\ln x}}$$

* Is f continuous on $[2,\infty)$? Yes.

(In fact, f is NOT continuous only at $x=1$ and $x=0$)

* Is f positive on $[2,\infty)$? Yes

* Is f decreasing on $[2, \infty)$? Yes

Since $x \cdot \sqrt{\ln x}$ is an increasing function (obvious),

$\dfrac{7}{x\sqrt{\ln x}}$ is decreasing.

* Consider $\displaystyle\int_2^\infty \frac{7}{x\sqrt{\ln x}}\,dx = \lim_{t\to\infty} \int_2^t \frac{7}{x\sqrt{\ln x}}\,dx$

$u = \ln x$. $du = \dfrac{1}{x}\,dx$. $\displaystyle\int \frac{7\,du}{\sqrt{u}} = \int 7u^{-1/2}\,du$

$= 7\dfrac{u^{1/2}}{1/2} = 14\sqrt{u}$

$\displaystyle\lim_{t\to\infty} 14\sqrt{\ln x}\Big|_2^t = \lim_{t\to\infty}\left(14\sqrt{\ln t} - 14\sqrt{\ln 2}\right) = \infty$

So, $\int_2^\infty \frac{7\,dx}{x\sqrt{\ln x}}$ diverges. By the integral test, the series

$$\sum_{n=2}^{\infty} \frac{7}{n\sqrt{\ln n}}$$ diverges.

Remainder Estimate from The Integral Test.

Suppose that we have used the integral test to determine that the series $\sum_{n=1}^{\infty} a_n$ converges.

Suppose that we then use partial sums to approximate the infinite sum of the series. Let say we use the $N^{\underline{th}}$ partial sum S_N. $S_N = \sum_{n=1}^{N} a_n$. We want to know how far away S_N is from the infinite sum?

The remainder of the approximation is

$$R_N = \sum_{n=1}^{\infty} a_n - S_N .$$

The integral test says that

$$\boxed{\int_{N+1}^{\infty} f(x)\,dx < R_N < \int_{N}^{\infty} f(x)\,dx}$$

E.g. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$. This is a p-series with $p=3$.

So it converges.

Suppose that we use S_{10} to approximate this infinite sum.

$$S_{10} = \sum_{n=1}^{10} \frac{1}{n^3} = 1 + \frac{1}{8} + \frac{1}{27} + \dots + \frac{1}{1000} \approx 1.1975$$

Q: Estimate the error?

$$R_{10} = \sum_{n=1}^{\infty} \frac{1}{n^3} - S_{10}.$$

Integral test says that

$$\int_{11}^{\infty} \frac{1}{x^3}\,dx < R_{10} < \int_{10}^{\infty} \frac{1}{x^3}\,dx$$

$$\int \frac{1}{x^3}\,dx = -\frac{1}{2x^2}.$$ So, $$\int_{10}^{\infty} \frac{1}{x^3}\,dx = \left(-\frac{1}{2x^2}\right)\Bigg|_{10}^{\infty} = \frac{1}{200} \simeq 0.005$$

$$\int_{11}^{\infty} \frac{1}{x^3}\,dx = \left(-\frac{1}{2x^2}\right)\Bigg|_{11}^{\infty} = \frac{1}{242} \simeq 0.00413$$

So,

$$\boxed{0.00413 < R_{10} < 0.005}$$

Q: Find the value of N such that S_N will estimate

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$ to within 0.001.

$$\int_{N+1}^{\infty} \frac{1}{x^3}\,dx < R_N < \int_{N}^{\infty} \frac{1}{x^3}\,dx < 0.001$$

$$-\frac{1}{2x^2}\Bigg|_{N}^{\infty}$$

$$= \frac{1}{2N^2} < 0.001$$

$$\frac{1}{2N^2} < 0.001$$

$$\frac{1}{N^2} < 0.002$$

$$N^2 > \frac{1}{0.002}$$

$$N > \sqrt{\frac{1}{0.002}} \approx 22.3607$$

$\Longrightarrow$ $N = 23$ is good enough.