## 5.4. Comparison Tests

### Direct Comparison Test

Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

* If $0 \leq a_n \leq b_n$ for all $n \geq N$, and $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

* If $0 \leq b_n \leq a_n$ for all $n \geq N$, and $\sum_{n=1}^{\infty} b_n$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

E.g. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ convergent or divergent ?

$n^2+1 > n^2$ for all $n \geq 1$.

$\dfrac{1}{n^2+1} < \dfrac{1}{n^2}$ for all $n \geq 1$.

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. Hence, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ must be convergent.

(by p-series, $p = 2 > 1$)

## More examples:

① $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+3n+1}$ ② $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n+1}$

③ $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$

Determine convergence/divergence of these series.

① $n^3+3n+1 > n^3$ for $n \geq 1$.

$$\frac{1}{n^3+3n+1} < \frac{1}{n^3} \text{ for } n \geq 1.$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges ($p$-series $p=3>1$), $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+3n+1}$ must converge.

② $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n+1}$

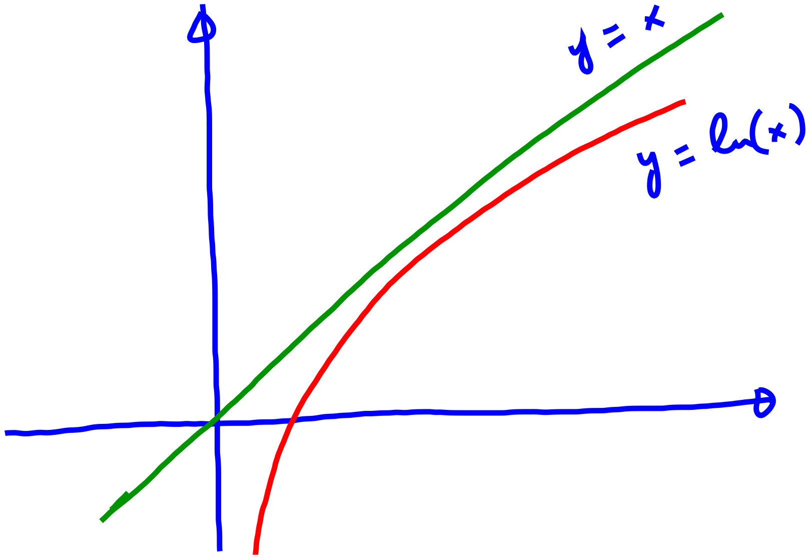
$2^n + 1 > 2^n$ for all $n \geq 1$.

$$\frac{1}{2^n+1} < \frac{1}{2^n}$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges (geometric series with common ratio $\frac{1}{2} < 1$),

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ must converge.

(3) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$




$\ln(n) < n$ for $n \geq 2$

$$\frac{1}{\ln(n)} > \frac{1}{n}.$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\ln(n)}$ must diverge.

($p$-series $p = 1$)

## Limit Comparison Test

Suppose that $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ are series with positive terms.

If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0$

( $L$ must be a finite number, $L > 0$ ) , then

$\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ behave the same way.

In other words, if one converges, the other converges. If one diverges, the other diverges.

E.g. $$\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{n^5+5}}$$

Note: when $n$ is large, this series seems to behave like

$$\sum_{n=1}^{\infty} \frac{2n^2}{\sqrt{n^5}} = \sum_{n=1}^{\infty} \frac{2n^2}{n^{5/2}} = \sum_{n=1}^{\infty} \frac{2}{n^{1/2}}$$

But we cannot directly compare the two series.

We will apply the limit comparison test for these 2 series:

$$\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{n^5+5}} \quad ; \quad \sum_{n=1}^{\infty} \frac{2}{n^{1/2}} \qquad \frac{2n^2+3n}{\sqrt{n^5+5}} \cdot \frac{n^{1/2}}{2}$$

$$a_n = \frac{2n^2+3n}{\sqrt{n^5+5}} \quad ; \quad b_n = \frac{2}{n^{1/2}}.$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\dfrac{2n^2+3n}{\sqrt{n^5+5}}}{\dfrac{2}{n^{1/2}}} = \lim_{n\to\infty} \frac{2n^{5/2}+3n^{3/2}}{2\sqrt{n^5+5}}$$

$$= 1 > 0$$

So, by the limit comparison test:

$$\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{n^5+5}}$$ behaves like $$\sum_{n=1}^{\infty} \frac{2}{n^{1/2}}.$$

Since $\sum_{n=1}^{\infty} \frac{2}{n^{1/2}}$ diverges, the original series diverges.

E.g.

$$\sum_{n=1}^{\infty} \frac{\ln\left(1+\frac{4}{n}\right)}{n} \qquad a_n = \frac{\ln\left(1+\frac{4}{n}\right)}{n}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \qquad b_n = \frac{1}{n^2}.$$

$$(\infty \cdot 0)$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\ln\left(1+\frac{4}{n}\right)}{n}}{\frac{1}{n^2}} = \lim_{n\to\infty} n \cdot \ln\left(1+\frac{4}{n}\right)$$

$$\frac{0}{0}$$

$$= \lim_{n\to\infty} \frac{\ln\left(1+\frac{4}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{1+\frac{4}{n}} \cdot \left(-\frac{4}{\cancel{n^2}}\right)}{\left(-\frac{1}{\cancel{n^2}}\right)}$$

$$= \lim_{n\to\infty} \left(\frac{4}{1+\frac{4}{n}}\right) = 4$$

Hence, by the limit comparison test, the original series behaves exactly like $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which is convergent

($p$ series, $p = 2 > 1$)

E.g. $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^2}$

$$\sin^2(n) \leq 1 \text{ for all } n \geq 1$$

$$\frac{\sin^2(n)}{n^2} \leq \frac{1}{n^2}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^2}$ converges by the direct comparison test.